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A VISUAL AND CONSTRUCTIVE WAY TO THE CONTRACTION MAPPING THEOREM

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Give the next series of exercises to senior undergraduate students, collect and evaluate the answeres and interesting experiences will be obtained.An important feature of these exercises that forming a guess for the solution the student can use computer experiments in a natural way.

Definition. *The mapping* $F:\mathbb{R}^2\to\mathbb{R}^2$ *is called* ***similarity*** *if there exists* $r\geq 0$ *such that*

$$d[F(x),F(y)]=r[x,y] \quad x,y\in\mathbb{R}^2.$$

A similarity map is also a ***contraction*** *if* $r<1$ *(fig.1).*

1. Give examples for similarity maps on the plane $\mathbb{R}^2$.

2. Could you find for **any** similarity $F:\mathbb{R}^2\to\mathbb{R}^2$ a closed disc $\boldsymbol{K}$ with

(*) $$F(\boldsymbol{K})\subset\boldsymbol{K}$$

where $F(\boldsymbol{K}):=\{F(x);\ x\in\boldsymbol{K}\}$?

Show, that if you find such a disc with center θ and radius R, then (*) is also satisfied for every disc $\boldsymbol{K}$ with the same center θ and radius greater then R.

3. Let

$F^{[1]}(\boldsymbol{K})=F(\boldsymbol{K})$ and $F^{[n]}(\boldsymbol{K})=F\cdot F^{[n-1]}(\boldsymbol{K})$ for $n=2,3,\ldots$.

Prove, that if (*) is satisfied, then also $F^{[n+1]}(\boldsymbol{K})\subset F^{[n]}(\boldsymbol{K})$.

Show that $F(\boldsymbol{K})$ is also a closed disc. What is the radius of $F^{[n]}(\boldsymbol{K})$?

Cantor axiom. *If* $\{\boldsymbol{D}_n;\, n=1,2,\ldots\}$ *is a sequence of bounded closed subsets of* $\mathbb{R}^2$ *with* $\boldsymbol{D}_n\subset\boldsymbol{D}_{n+1}$, *then* $\bigcap_{n=1}^{\infty}\boldsymbol{D}_n$ *is not empty if* (*) *is satisfied.*

4. Based on Ex.3 and the Cantor axiom, $\bigcap_{n=1}^{\infty}F^{[n]}(\boldsymbol{K})$ is not empty. Let

$$p\in\bigcap_{n=1}^{\infty}F^{[n]}(\boldsymbol{K})\,;$$

what is the connection between $\{F^{[n]}(\boldsymbol{K})\,;\, n=1,2,\ldots\}$ and p ?

5. Give a fixed origin θ on the plane $\mathbb{R}^2$. Then a rotation by ϕ and **scaling** by r are well defined.Let F be the composition of a rotation, scaling and a shift by the vector b.

a/ Using drawing algorithm on a personal computer,visualize the sequence $\{F^{[n]}(\boldsymbol{B})\,;\, n=1,2,\ldots\}$ with different starting figures (bounded closed) $\boldsymbol{B}$.

b/ Giving the matrix form of F, $F^{[2]},\ldots$ compute $\lim F^{[n]}(x)$ for different $x\in\mathbb{R}^2$.

Compare the results a/ and b/.

6. Let $x \in \mathbb{R}^2$, D be a closed bounded subset of $\mathbb{R}^2$ and p be as in Ex.4. and $r<1$.

a/ What is the connection between $\{F^{[n]}(x);\ n=1,2,\dots\}$ and p.

b/ What is the connection between $\{F^{[n]}(D);\ n=1,2,\dots\}$ and p.

Definition. *The mapping* $F:\mathbb{R}^2 \to \mathbb{R}^2$ *is called contraction if*

$$d[F(x),\ F(y)] \leq r[x,y] \quad x,y \in \mathbb{R}^2$$

for $0 \leq r < 1$.

7. Prove the Contraction Mapping Theorem:

If F is a contraction, then $\{F^{[n]}(x);\ n=1,2,\dots\}$ *is convergent for every* $x \in \mathbb{R}^2$. *Moreover, the limit is the same for every* $x \in \mathbb{R}^2$.

SOLUTIONS

2. *Fig.2* shows the simplest solution: (*) is satisfied for K iff for the radius R of K

(R) $$d[\theta,\ F(\theta)] + rR < R\,,$$

where θ is the center of K.

If (R) does not come in by traditional geometry quickly, then it is useful to use experiments by computer graphics for developing a guess like (R). In this later case a document of these experiments is a part of the solution.

3. If $B \subset C$ then $F(B) \subset F(C)$ for any mapping F and hence

(*)

imply

$$F^{[n+1]}(K) \subset F^{[n]}(K) \quad \text{for} \quad n=1,2,\dots\,.$$

$x \in K$ iff $d[\theta, x] \leq R$. Hence, it follows from the *Definition* that $y \in F(K)$ iff $d[F(\theta, y] \leq rR$. It follows that the radius of $F^{[n]}(K)$ is $r^n R$.

4. There is exactly one point p contained in every $F^{[n]}(K)$. p has the following property.

(P) Let $K_{\epsilon,p}$ *be the disc with center* p *and radius* ϵ*. Then for 7every* $\epsilon>0$ *there is* N *such that*

$$F^{[n]}(K)\subset K_{\epsilon,p}$$

if $N>n$.

Compare *(P)* with the next statement:

"*The sequence* $\{F^{[n]}(x);\ n=1,2,\ldots\}$ *converges to* p *for every* $x\in\mathbb{R}^2$ "

5. This is the fundamental experiment of this material. This is a visual and numerical exploration of the Contraction Mapping Principle, however we use only similarity for **F**. The exploration may give also ideas for a constructive proof of the theorem.

6. Since for any $x\in\mathbb{R}^2$ we have K with $F(K)\subset K$ and $x\in K$, $F^{[n]}(x)\Rightarrow p$ of Ex.4. for every $x\in\mathbb{R}^2$.

For every such D we have K with $D\subset K$ and $F(K)\subset K$. Hence **(P)** holds also for $F^{[n]}(D)$.

7. See the foregoing exercises and figures 2-4.

Some educational experiences

To show that $F^{[n+1]}(K)\subset F^{[n]}(K)$ for $n=1,2,\ldots$ it was rather difficult most of the students since they did not recognized that $B\subset C$ implies $F(B)\subset F(C)$ for any mapping F. Hence, it was exploited in almost every solution that K is a disc and F is a similarity and hence $F(K)$ is also disc.

The difference between **(P)** and the "ordinary" convergence of a point sequence was skipped over in almost every solution of Ex.4 resp. Ex.6

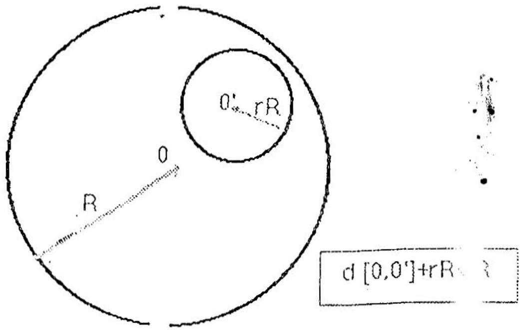

fig. 2

fig. 2+ fig. 3:

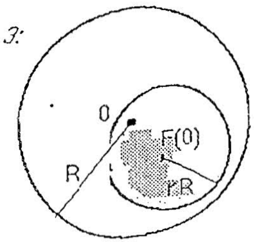

fig. 4

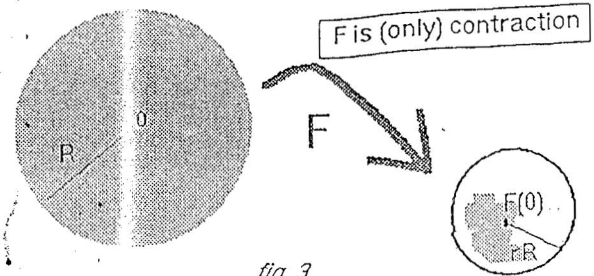

fig. 3

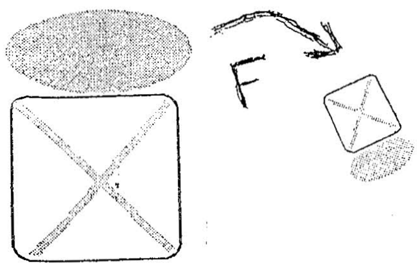

fig.1

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Abstract. This series of exercises is addressed to senior undergraduate students. An important feature of these exercises that forming a guess for the solution the student can use computer experiments in a natural way.

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