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THE VALUE OF PROOFS AND INTUITIVE EXAMPLES IN TEACHING MATHEMATICS

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1 . To conceive this article, we started from the following general remark : in many places, the higher mathematical teaching is very formal, not motivated and, sometimes, completely separated from the inner reality of living mathematical objects. We have noticed with anxiety that this unnatural way of teaching Math was successful in destroying all its real and normal connections with other sciences. From our point of view, we have to start with simple, but essential examples, to work gradually with them in order to discover the ideas which lead to the proof.

We consider examples as being not only mere numerical verification of formulas, but everything else which may motivate or explain the elaborated and sophisticated theory.

S.Mac Lane ([1])is right when he states that the proof is not a beginning, but an end of a very complex intellectual process consisting of some important stages : intuition,choice,error,speculation, conjecture, ..., and proof. We are going to give two examples in which, firstly, we'll try to explain the abstract notion of a "dimension" of a vector space using the intuitive notion of "*degrees of freedom*" for an object, and secondly, we'll construct the whole theory of vector subspaces in $\mathbb{R}^n$ and the theory of linear system started from Gauss's elimination method and from other elementary notions in vector spaces.

2 . If we want to describe the vectors in a vector space , we usually give a generic object in this space as a function of some parameters $x_1, x_2, ..., x_n$: $f(x_1, x_2, ..., x_n)$. If these parameters are linearly independent, i. e. none of them is a linear combination of the others, and if them is a linear combination of $x_1, x_2, ..., x_n$, i. e. $f(x_1, x_2, ..., x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_1, ..., a_n$ are fixed vectors, we say that the set of vectors $\{ f(x_1, ..., x_n) \mid x_1, ..., x_n \in \mathbb{R} \}$ has *n degrees of freedom* and $\{ a_1, ..., a_n \}$ is a *basis* in this set. For instance $V = \left\{ \begin{pmatrix} x_1 - 2x_2 & 2x_1 \\ x_2 & 2x_1 + 3x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$ is a vector space with 2 degrees of freedom. Because $\begin{pmatrix} x_1 - 2x_2 & 2x_1 \\ x_2 & 2x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} x_1 + \begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix} x_2$, $\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix} \right\}$ is a basis in V. We call the number of degrees of freedom, the *dimension* of the vector space. So that the dimension of V is 2.

But, if we consider $W = \left\{ \begin{pmatrix} x_1 - 2x_2 & 2x_1 \\ x_2 & 2x_1 + 3x_2 \end{pmatrix} \mid x_1 - 2x_2 = 0,\ x_1, x_2 \in \mathbf{R} \right\}$, the dimension of W is not 2 because x_1 and x_2 are not free through $\mathbf{R}$, they are dependent, namely $x_1 = 2x_2$. If we substitute x_1 with $2x_2$ in the definition of a generic object in W, $\begin{pmatrix} x_1 - 2x_2 & 2x_1 \\ x_2 & 2x_1 + 3x_2 \end{pmatrix}$ we obtain $\begin{pmatrix} 0 & 4x_2 \\ x_2 & 7x_2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 1 & 7 \end{pmatrix} x_2$, where x_2 is free through $\mathbf{R}$. So the dimension of W is one. Remark that the relation $x_1 - 2x_2 = 0$ is linear. If this last relation would not be a linear one the results above were not true.For instance,if $x_1^2 - 2x_2 = 0$, $x_1 = \pm\sqrt{2x_2}$ and x_2 is not free in $\mathbf{R}$ (it must be positive or zero). Here we consider x_1 free in $\mathbf{R}$ and $x_2 = \frac{x_1^2}{2}$. We obtain $\begin{pmatrix} x_1 - 2x_2 & 2x_1 \\ x_2 & 2x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_1^2 & 2x_1 \\ \frac{x_1^2}{2} & 2x_1 + \frac{3}{2}x_1^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} x_1 - \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} x_1^2$. Can we say that W is a vector space of dimension 2 ? No ! W is not a vector space at all ! Even we substitute x_1^2 with x_1^3, W is not a vector space with the usual addition of matrix.

Example 1 .

$S = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1 - 2x_2 + x_3 = 0, x_2 + 2x_4 = 0, x_1, x_2, x_3, x_4 \in \mathbf{R}\}$ is a vector space with 2 degrees of freedom, so that the dimension of S is 2.Indeed,the two relations are "independent" and we can express x_1 and x_2 as linear functions of x_3 and x_4, etc.

The two notions, the *number of degrees of freedom* and *the dimension* have the same significance. But, from our point of view, the *former* is more suggestive and more motivated (for engineers, for instance)than the latter.

3. Let n be a fixed natural number, $n \geq 2$, and $\mathbf{R}^n$ the n-dimensional vector space over the field of real numbers.

Definition 1 .

A system of k vectors $\mathbf{a}_1 = (a_{11}, a_{12}, ..., a_{1n}), \mathbf{a}_2 = (a_{21}, a_{22}, ..., a_{2n}), ..., \mathbf{a}_k = (a_{k1}, a_{k2}, ..., a_{kn})$, in $\mathbf{R}^n$ is called a *Gauss system if for every* $i = 1, 2, ..., k-1$, we have $z(a_i) < z(a_{i+1})$, where $z(a_i) = \min\{ j \mid a_{ij} \neq 0\}$.

Example 2 . $\{(0, 1, -1, 1, 2), (0, 0, 2, 0, 1), (0, 0, 0, 0, 3)\}$ is a Gauss system in $\mathbf{R}^5$.

Proposition 1 . Every Gauss system $\{\mathbf{a}_1, ..., \mathbf{a}_k\}$ is a linear independent system, so that $\dim Sp\{\mathbf{a}_1, ..., \mathbf{a}_k\} = k \leq n$, where $Sp\{\mathbf{a}_1, ..., \mathbf{a}_k\}$ is the smallest linear subspace in $\mathbf{R}^n$ generated by $\mathbf{a}_1, ..., \mathbf{a}_k$. So that if $\{\mathbf{a}_1, ..., \mathbf{a}_k\}$ is a Gauss system then $\{\mathbf{a}_1, ..., \mathbf{a}_k\}$ is a basis in $Sp\{\mathbf{a}_1, ..., \mathbf{a}_k\}$.

Theorem 2. Every non zero vector subspace $S = Sp\{\mathbf{b}_1, ..., \mathbf{b}_t\}$ in $\mathbf{R}^n$ can be generated by a Gauss system of vectors.

Proof . First of all we may change $\mathbf{b}_1, ..., \mathbf{b}_t$ between them such that $z(\mathbf{b}_1) \leq z(\mathbf{b}_2) \leq \cdots \leq z(\mathbf{b}_t)$. If $z(\mathbf{b}_1) = z(\mathbf{b}_2) = j$, $\mathbf{b}_1 = (0, ..., 0, b_{1j}, b_{1j+1}, ..., b_{1n})$, $\mathbf{b}_2 = (0, ..., 0, b_{2j}, b_{2j+1}, ..., b_{2n})$, let change $\mathbf{b}_2$ with $\mathbf{b}_2' = \left(0, ..., 0, 0, b'_{2j+1}, ..., b'_{2n}\right)$, where $\mathbf{b}_2' = b_{1j} \cdot \mathbf{b}_2 - b_{2j} \cdot \mathbf{b}_1$. It is easy to see that $\{\mathbf{b}_1, \mathbf{b}_2', \mathbf{b}_3, ..., \mathbf{b}_t\}$ is again a generating system in S. But now $z(\mathbf{b}_1) < z(\mathbf{b}'_2)$.We repeat the same reasoning (Gauss elimination in fact)and change $\mathbf{b}_3$ with $\mathbf{b}'_3$ if $z(\mathbf{b}_1) = z(\mathbf{b}_3)$, let $\mathbf{b}'_3 = \mathbf{b}_3$ if $z(\mathbf{b}_1) = z(\mathbf{b}_3)$, etc. Finally we'll have a new system of generators $\{\mathbf{b}_1, \mathbf{b}'_2, \mathbf{b}'_3, ..., \mathbf{b}'_t\}$ such that $z(\mathbf{b}_1) < z(\mathbf{b}'_2) \leq z(\mathbf{b}'_3) \leq \cdots \leq z(\mathbf{b}'_t)$. Now, if $z(\mathbf{b}'_2) = z(\mathbf{b}'_3)$,let change $\mathbf{b}'_3$ with $\mathbf{b}''_3$, $\mathbf{b}'_4$ with $\mathbf{b}''_4$,...,$\mathbf{b}'_t$ with $\mathbf{b}''_t$, such that $z(\mathbf{b}'_2) < z(\mathbf{b}''_3) \leq z(\mathbf{b}''_4) \leq \cdots \leq z(\mathbf{b}''_t)$. We repeat the same reasoning and obtain finally a new system of generators for S, $\{\mathbf{b}_1, \mathbf{b}'_2, \mathbf{b}''_3, ...\}$, such as $z(\mathbf{b}_1) < z(\mathbf{b}'_2) < z(\mathbf{b}''_3) < \cdots$.If some of the vectors $\mathbf{b}'_k$, $\mathbf{b}''_k$, etc., are zero-vectors, we drop them out of the sequence of generators.Therefore, in the end, the number of generators may decrease.

Corollary 3 (the rank theorem). If $\mathbf{b}_i = (b_{i1}, b_{i2}, ..., b_{in})$, $i = 1, 2, ..., k$, are k vectors in $\mathbf{R}^n$ and $B = (b_{ij})$, $i = 1, 2, ..., k$; $j = 1, 2, ..., n$, then *rank* $B = \dim Sp\{\mathbf{b}_1, ..., \mathbf{b}_k\}$.

Proof . All the changes performed during the proof of the Theorem 2 mean left multiplications of the matrix B with invertible matrix. Therefore the rank of B do not change when we change the system of generators of the vector subspace $Sp\{\mathbf{b}_1, ..., \mathbf{b}_k\}$. Hence everything reduces to the case of a Gauss system of generators when everything becomes clear.

We can continue in developing the theory of linear systems, etc. in a very easy and natural manner, having as a fundamental result Theorem 2.We presented an example of a very elementary and natural concept (a Gauss system of generators)and its applications to more sophisticated things from linear algebra. This last concept has a very extended intuitive power all over abstract proofs of many results on finite vector spaces.In our opinion,this is an intuitive example.

Is it possible to teach mathematics without such examples ? We think that it is not.

References

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