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A GENERALIZATION OF CAUCHY'S THEOREM

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Some generalizations of Cauchy's theorem appears in many journals (see Gazeta Matematică). The aim of this note is to prove a generalization of this theorem.

Theorem. *Let $f_1, f_2, \ldots, f_n, f_{n+1}, f_{n+2}$ be real functions defined in interval $[a,b]$ (i.e. $f_i : I \to \mathbb{R}$, where i runs through $\{1,2,\ldots,n+2\}$ and $I=[a,b]\subset\mathbb{R}$) and let $a=x_0<x_1<\ldots<x_n=b$ be points.*

The following conditions:

1) f_i are continuous real functions where i runs through $\{1,2,\ldots,n+2\}$;

2) for every i who runs through $\{1,2,\ldots,n+2\}$ there exist the n-th derivative of f_i on the interval (a,b) and the derivatives $f_i', f_i'', \ldots, f_i^{(n-1)}$ are continuous on I;

imply that there exist $c\in(a,b)$ such that

$$\begin{vmatrix} f_1^{(n)}(c) & f_2^{(n)}(c) & \ldots & f_{n+2}^{(n)}(c) \\ f_1(x_0) & f_2(x_0) & \ldots & f_{n+2}(x_0) \\ f_1(x_1) & f_2(x_1) & \ldots & f_{n+2}(x_1) \\ \cdot & \cdot & \ldots & \cdot \\ f_1(x_n) & f_2(x_n) & \ldots & f_{n+2}(x_n) \end{vmatrix} = 0$$

Proof. Set $F(x)=\begin{vmatrix} f_1(x) & f_2(x) & \dots & f_{n+2}(x) \\ f_1(x_0) & f_2(x_0) & \dots & f_{n+2}(x_0) \\ f_1(x_1) & f_2(x_1) & \dots & f_{n+2}(x_1) \\ \cdot & \cdot & \dots & \cdot \\ f_1(x_n) & f_2(x_n) & \dots & f_{n+2}(x_n) \end{vmatrix}$, where $F: I\to\mathbb{R}$.

We deduce:

1) $F(x_0)=F(x_1)=\dots=F(x_n)=0$;

2) On every (x_i, x_{i+1}) F satisfy the conditions of Rolle's Theorem where i runs through $\{0,1,\dots,n-1\}$. We obtain by applying Rolle's Theorem that there exist $k_i\in(x_i, x_{i+1})$ such that $F'(k_i)=0$, where i runs through $\{0,1,\dots,n-1\}$;

3) On every (k_i, k_{i+1}) F' satisfy the conditions of Rolle's Theorem where i runs through $\{0,1,\dots,n-2\}$. We obtain by applying Rolle's Theorem that there exist $k_i^1\in(k_i, k_{i+1})$ such that $F''(k_i^1)=0$, where i runs through $\{0,1,\dots,n-2\}$;

4) Proceeding similarly we can demonstrate (by applying Rolle's Theorem) that there exist $c\in(a,b)$ such that $F^{(n)}(c)=0$.

Notes. 1°) For $f_1(x)=x$, $f_i(x)=x^{n+1-i}$, where i runs through $\{2,3,\dots,n+1\}$ and $f_{n+2}(x)=1$ we obtain the third problem from [3] (see 79'th page);

2°) For $f_1(x)=f(x)$, $f_2(x)=g(x)$, $f_i(x)=x^{n+1-i}$ where i runs through $\{3,4,\dots,n+1\}$ and $f_{n+2}(x)=1$ we obtain a theorem which was proved in [2] (without third condition: $g^{(n)}(c)\neq 0$, $\forall c\in(a,b)$);

3°) For $n=1$ we obtain the 9'th problem from [1] (page 3): " If $f_i(x)$ are continuous on $I:=[a,b]$ and there exist first derivative of f_i on (a,b), where i runs through $\{1,2,3\}$, then there exist $c\in(a,b)$ such that $\begin{vmatrix} f_1'(c) & f_2'(c) & f_3'(c) \\ f_1(a) & f_2(a) & f_3(a) \\ f_1(b) & f_2(b) & f_3(b) \end{vmatrix}=0$.

3°.1. Setting $f_1(x)=\frac{x}{f(x)}$, $f_2(x)=\frac{1}{f(x)}$ and $f_3(x)=1$ we obtain a theorem from [4];

3°.2. Setting $f_1(x)=\frac{f(x)}{x-d}$, $f_2(x)=\frac{1}{x-d}$ and $f_3(x)=1$ we reobtain a theoreme from [5];

3°.3. Setting $f_1(x)=\frac{f(x)}{x}$, $f_2(x)=\frac{1}{x}$ and $f_3(x)=1$ we reobtain D. Pompeiu's Theorem from [6].

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Abstract. The aim of this note is to prove a generalization of Cauchy's theorem.

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