Lucrările Seminarului de
CREATIVITATE MATEMATICĂ
Vol. 7 (1997-1998), 65-76

THE CONCEPT OF TRANSFORMATION AND ITS MEANING IN GEOMETRY

V. NTZIAHRISTOS

1. Introduction

There are generally two ways of developing Modern Geometry.

a) The **synthetical**, where we start by defining the principal concepts, the axioms, the definitions referring to the principal concepts, the theorems e.t.c. This way was settled by the ancient Greek geometres and especially by Euclid in the "Elements", marks out the "abstract" nature of Geometry and has been successfully used in all mathematical fields as well as in Logic and in Philosophy, having no substantial success as far as Philosophy is concerned.

b) The **analytical**, where we start by constructing a model on the plane, the sphere or another object. Then from the study and the calculations of the different parts of the model, we draw our conclusions about the Geometry which is expressed by this model. This approach emphasizes the interdependence of Geometry with other mathematical fields and the "real world" of the model. This way is called analytical, because we use coordinates for the study of the model.

Regarding the synthetical way of development of Geometry, F. Klein inaugurated the study of Geometry with the help of transformations of Euclidean plane in 1872, after the announcement of Erlanger program. Furthermore, as complex numbers are the most convenient algebraic tool in order to express and study the transformations, we' ll mention here their basic properties.

2. Complex numbers C

The concept of complex number is an algebraic concept having substantial geometrical meaning, which can be expressed with the help of cartesian coordinates.

2.1. Definition

Complex number is a point $z = (\alpha, b)$ of Cartesian plane, where the x-axis (real) is measured by the usual real unit, while the y-axis (imaginary) is measured by a different unit i. The real number x is called real part of z ($x = \mathrm{Re}(z)$) and the real number y is called imaginary part of z ($y = \mathrm{Im}(z)$).

2.2. Operations on complex numbers

Being points of a plane, the complex numbers can be combined with two operations very common in vector analysis : the vector addition and the scalar multiplication of a vector by a real number. The operations can be defined algebraically, geometrically and vectorially as follows :

2.2.1. Algebraically

Let $z = (\alpha, b)$, $z_1 = (\alpha_1, b_1)$ and $z_2 = (\alpha_2, b_2)$ be three complex numbers. We define as **sum** of z_1 and z_2 the complex number $z_1 + z_2 = (\alpha_1 + \alpha_2, b_1 + b_2)$. The multiplication of a real number k by the complex number z is the number $\kappa \cdot z = (\kappa\alpha, \kappa b)$. That is, the multiplication of a real number k by the complex number $z = (\alpha, b)$ can be performed, by multiplying both the real and the imaginary part of z by k.

2.2.2. Geometrically

The addition and the scalar multiplication are shown in figures *1* and 2 respectively.

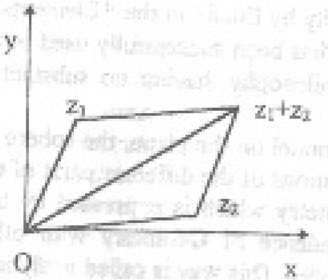


Figure 1

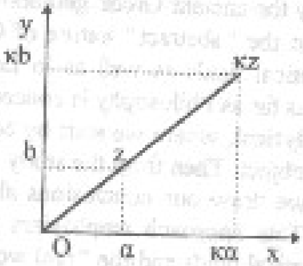


Figure 2

Geometrically, the result of the multiplication of a number k by the complex $z = (\alpha, b)$ is the complex k z , where the points O, z, kz are collinear and the distance of kz from the origin O is k times the distance of z from the origin.

2.2.3. Vectorially

By using vectorial concepts z can be written : $z = \alpha\,(1,0) + b\,(0,1) = \alpha + ib$ and $kz = k\alpha\,(1,0) + kb\,(0,1) = k\alpha + ikb$

The form $\alpha + ib$ is called Cartesian form of z.

2.3. Concepts relating to the complex numbers

A useful geometrical concept is the modulus $|z|$ of the complex number. Algebraically the modulus of the complex number $z = \alpha + ib$ is $|z| = \sqrt{\alpha^2 + b^2}$, while geometrically (fig. 3) the modulus $|z|$ is the distance of the point z from the origin O.

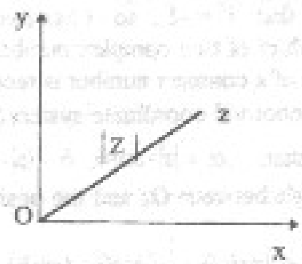


Figure 3

Concepts relating to the modulus are the following.

Let $z_1 = \alpha_1 + ib_1$ and $z_2 = \alpha_2 + ib_2$ be two complex numbers. The distance between the points z_1 and z_2 is given algebraically by the relationship $|z_1 - z_2| = \sqrt{(\alpha_1 - \alpha_2)^2 + (b_1 - b_2)^2}$, while geometrically is shown in figure 4.

Conjugate of the complex number $z = \alpha + ib$ is the number $\bar{z} = \alpha - ib$. The point $\bar{z} = \alpha - ib$ is the symmetrical of z with respect to the x-axis (fig. 5).

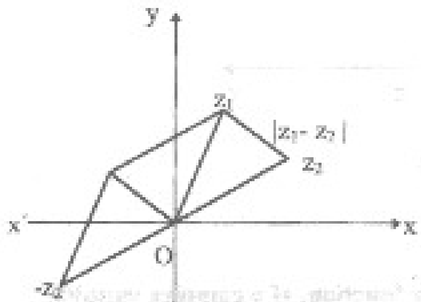


Figure 4

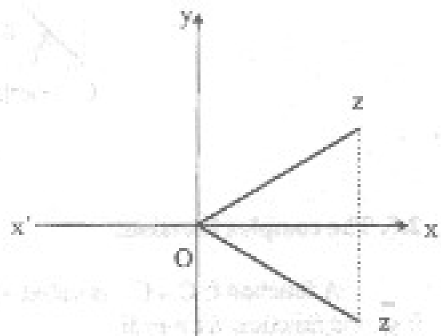


Figure 5

2.4. Multiplication of complex numbers

Let $z_1 = \alpha + ib$ and $z_2 = c + id$ be two complex numbers. Their product is defined as $z_1 \cdot z_2 = (\alpha + ib)(c + id) = (\alpha c - bd) + i(\alpha d + bc)$

E.g. $ii = i^2 = (0+1i)(0+1i) = -1$

$(1+i)(1+i) = (1+i)^2 = (1-1) + i(1+1) = 2i$ e.t.c.

The multiplication in C has the same properties with the multiplication in R, that is:

1. Commutative $z_1 \cdot z_2 = z_2 \cdot z_1$
2. Associative $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$
3. Distributive to the addition $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

4. For every element $z \in C$ where $z \neq 0+i0=0$ there is such a $z' \in C$, so that $z \cdot z' = 1 + i0 = 1$.

The main new element is that $i^2 = -1$, so -1 has two square roots.

In order to study the product of two complex numbers better, the introduction of the trigonometric (polar) form of a complex number is recommended.

Let $z=(\alpha,b)$ be in an orthonormal coordinate system (fig. 6). If $(Oz) = |z|$, from the right triangle Ozz′ we take that: $\alpha = |z| \cdot \cos\theta$, $b = |z| \cdot \sin\theta$ where $\theta (0 \leq \theta \langle 2\pi)$ the argument of z, that is the angle between Oz and the positive semi-axis Ox and we denote arg (z)= θ (mod 2π).

Then $z = \alpha + ib = |z| \cdot \cos\theta + i|z| \sin\theta = |z|(\cos\theta + i\sin\theta)$ (1)

(1) is called trigonometric form of z.

Apart from the trigonometric form, we often use the polar form of z, which is $z = |z| e^{i\theta}$

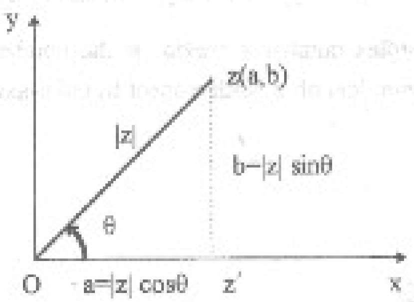


Figure 6

2.5. The complex functions

A function f: C→C is called a **complex function, of a complex variable**. E.g. the function f(z)=z+3i.

The complex exponential function is defined by substituting e^z for $e^{a+ib} = e^a$ (cos b + i sin b), so that $e^{i\theta} = \cos\theta + i \cdot \sin\theta$ (Formula Euler). The main reason for adopting this definition is the basic property of exponential functions $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$ hence :

$$e^{i\theta} e^{i\varphi} = (\cos\theta + i\sin\theta) \cdot (\cos\varphi + i\sin\varphi) = \cos(\theta+\varphi) + i\sin(\theta+\varphi) = e^{i(\theta+\varphi)}$$

The multiplication of complex numbers in a polar form is defined as follows

Let $z_1 = |z_1| e^{i\theta}$ and $z_2 = |z_2| \cdot e^{i\varphi}$, then $z_1 \cdot z_2 = |z_1||z_2| \cdot e^{i(\theta+\varphi)}$, that is rotates the image of z_1 by the angle φ.

As follows from the above, the complex numbers are essentially points of the cartesian plane. The complex numbers can be combined algebraically, with the help of addition and multiplication. These operations, as we' ve seen, have also a geometrical meaning in relation to the distances, the angles and generally the motion on the plane.

2.6. Transformations

A special case of the complex function is the transformation. A transformation is an one - to - one function $f: A \to A$ where $A \subset R^2$ or $A \subset R^3$, $A \subset C^2$ e.t.c.

We classify the transformations in two categories: the **Euclidean** and the **non - Euclidean.**

Euclidean are the transformations which preserve the sides and the angles of the figures. Such transformations (which are also called isometric) are the translation, rotation, axial symmetry e.t.c.

Non - Euclidean transformations are homothety , the inversion e.t.c.

The function $f: C \to C$ where $f(z) = z+(2+3i)$, which adds to the complex number z the complex number 2+3i is a translation which transposes the points of the complex plane by a constant vector $\vec{u} = 2\vec{i} + 3\vec{j}$ (fig. 7).
The translation has the form $f(z) = z+k$, $k \in C$.

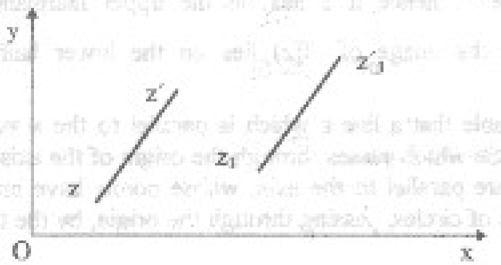


Figure 7

The **rotation** is a transformation of the form $f(z) = e^{i\theta} z$, where $\theta \in R$ is the angle of rotation (fig. 8).

The homothety is a transformation of the form $f(z) = kz$, with $k \in R$. If $\kappa > 1$ is called **magnification**, but if $\kappa < 1$ **diminution** (fig. 9).

The inversion is a transformation $f: C^* \to C$ where $f(z) = \frac{1}{z}$. We mention that the geometrical meaning of the transformation of the inversion is very important and so should be examined in details.

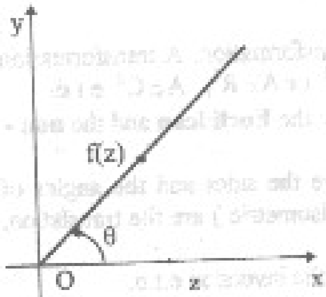


Figure 8

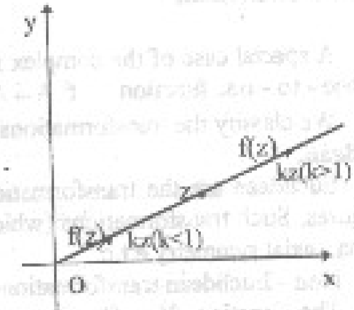
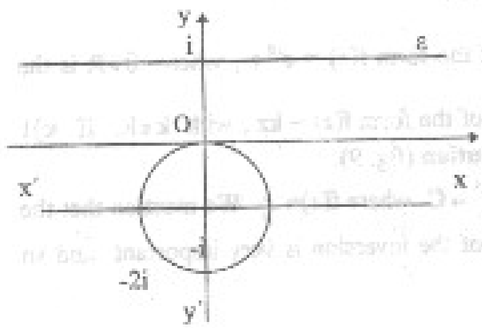


Figure 9

The inversion is a non - Euclidean transformation . We first observe that if the point z lies in the interior of the unit circle $x^2+y^2=1$ (that is $|z|\langle 1$) , then $|f(z)| = \left|\frac{1}{z}\right| \rangle 1$, i.e. the image of z lies in the exterior of the circle. Yet, is if $z = r \cdot e^{i\theta}$ then $f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$, hence if z lies on the upper halfplane of x-axis (that if $0 \le \theta \le \pi$) , then the image of f(z) lies on the lower halfplane of the x-axis ($-\pi \le \theta \le 0$).

It is remarkable that a line ε which is parallel to the x-axis is mapped by the inversion onto a circle which passes through the origin of the axis (fig. 10) Hence, the net of lines which are parallel to the axis, whose points have integer coordinates, is transformed in a net of circles, passing through the origin, by the transformation of the inversion.



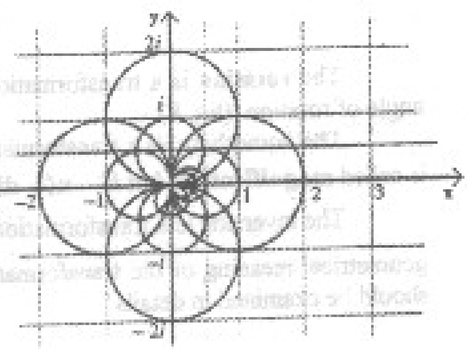


Figure 10

Figure 11

Indeed, let $z = x + i\alpha$ be a point of the straight line $y = \alpha$ which is parallel to the x-axis. Then $f(z) = \frac{1}{z} = \frac{1}{x+i\alpha} = \frac{x-i\alpha}{x^2+\alpha^2} = k + il$, where $k = \frac{x}{x^2+\alpha^2}$ and $l = \frac{-\alpha}{x^2+\alpha^2}$.

3. Conformal Mappings

3.1. Conformal Transformations

A transformation f is called **conformal**, if it preserves the angles, that is the angles of a figure S are equals to the angles of the figure $S_1=f(S)$. All the transformations examined above are conformal. E.g. the translations, the rotations and the homotheties.

The proof of this fact is simple and derives directly from the definition of the translation, the rotation and the homothety. We shall prove that the inversion is also a conformal transformation under certain conditions.

3.1.1. Theorem

The inversion about every point of the plane excepting the origin is a conformal transformation.

Proof

Let $z(z \neq 0)$ be a point of the complex plane and c_1, c_2 two curves passing through the point z (fig. 12).

Yet, let c_1', c_2' be the images of c_1, c_2 through an inversion f. Then c_1, c_2 pass through the point $z' = \frac{1}{z}$.

Let α, b, be points of c_1, c_2 so that O, α, b are collinear. Let $\alpha' = \frac{1}{\alpha}$ and $b' = \frac{1}{b}$ be the images of a,b respectively. Then $\alpha z b = arg\left(\frac{b-\alpha}{\alpha - z}\right)$ and

$$\alpha'\hat{z}'b' = arg\left(\frac{b'-z'}{\alpha'-z'}\right) = arg\left(\frac{\frac{1}{b}-\frac{1}{z}}{\frac{1}{\alpha}-\frac{1}{z}}\right) = arg\left(\frac{\frac{z-b}{bz}}{\frac{z-\alpha}{\alpha z}}\right) = arg\left(\frac{\alpha(b-z)}{b(\alpha-z)}\right).$$

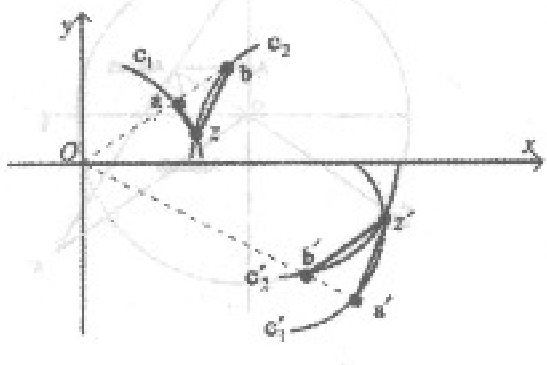


Figure 12

From the fact that α, O, b are collinear, derives that $\alpha \hat{z} b = \alpha' \hat{z'} b'$. Indeed, by assuming that a,b tend to z (since the inversion is a continuous function), α', b' tend to z', too. As a consequence, the angle between the tangents to c_1, c_2 in z is equal to the angle between c_1', c_2' in z' - that is the inversion is a conformal transformation.

3.2 Stereographic projection

The stereographic projection is not a transformation, as it is a mapping of the surface of the sphere onto the complex plane and not of the plane onto the plane. Howerer, it has all the characteristics of a transformation and will be examined here.

Without loss of generality we consider the unit sphere on the space R^3, given by the equation $x^2+y^2+z^2=1$. This sphere has centre the origin O (0,0,0). The point P(0,0,1) (fig.13) is called pole (centre) of the stereographic projection.

Let A (a,b,c) a point of the sphere $(A \neq P)$. The halfline PA intersects the plane xOy on the point A′ which is the image of A. We observe that all the points of the sphere excepting the pole P, are mapped onto points of the complex plane xOy. As we mentioned, the pole P is not mapped onto xOy, since the line PA is not uniquely defined.

The stereographic projection is very interesting, since it connects directly geometrical properties of the sphere and the plane. These properties are used by the cartographers in the constructions of maps. E.g. the stereographic projection preserves the angles, that is, it is a conformal mapping as we shall see below. This fact means that the figures of the geometric objects are preserved when mapped on a paper. But the stereographic projection is also very important for the non- Euclidean geometries, as the mapping of the surface of the sphere onto the plane allows us to study properties of a non- Euclidean Geometry like Spherical Geometry on the plane, in a simpler way.

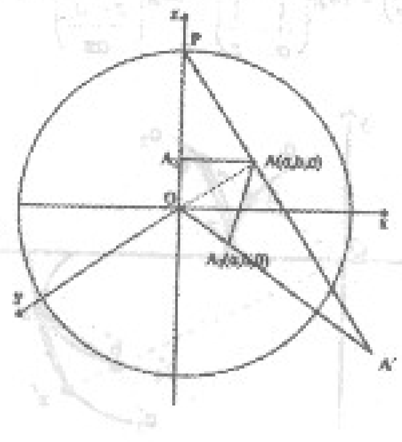


Figure 13

A basic problem is the calculation of the position of the point f(A)=A΄ (fig.13), that is, the calculation of the coordinates of the point A΄. As the right triangles A΄OP and AA ${}_2$P as well as the triangles A΄OP, A΄A ${}_1$A are similar, we' ll have

$$\frac{OP}{PA_2} = \frac{OA'}{AA_2} = \frac{A'O}{OA_1}$$

Let $\frac{OP}{OA_2} = l$. But $\frac{OP}{PA_2} = \frac{1}{1-c}$, hence $l = \frac{1}{1-c}$

Consequently $A'(x_o, y_o, z_o) = \frac{1}{1-c}(\alpha, b, 0) \cong \frac{\alpha + ib}{1-c}$.

Here, we consider the plane xOy as a complex plane, hence the image of the surface of the unit sphere is mapped on the plane, so that the point A(a,b,c) has as its image the complex number x+iy with $x = \frac{\alpha}{1-c}$, $y = \frac{b}{1-c}$.

Like the inversion, the stereographic projection maps the circles onto lines. Figure 14 shows the image of a circle of the sphere, which passes through the pole P and it is the intersection of the circle' s plane with the plane xOy.

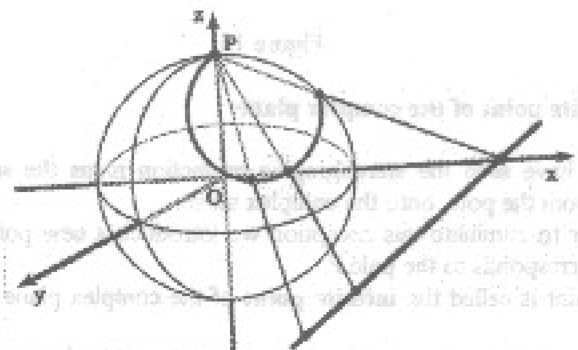


Figure 14

3.2.1 Theorem

The stereographic projection is a conformal mapping.

Proof

Without loss of generality we shall prove the proposition, when the images are straight lines by assuming that the straight lines AB, AC are the images of two circular arcs which pass through P and intersect on N (fig. 15). Due of the symmetry, the angle between the arcs on P equals to the angle between the arcs on N. The angle between the arcs on P equals, by definition, to the angle between the tangents of the circles (and the spheres) on P.

As the tangents ε_1, ε_2 are parallels to AB, AC obviously $\varepsilon_1\hat{P}\varepsilon_2 = B\hat{A}C$, that is the stereographic projection is a conformal mapping.

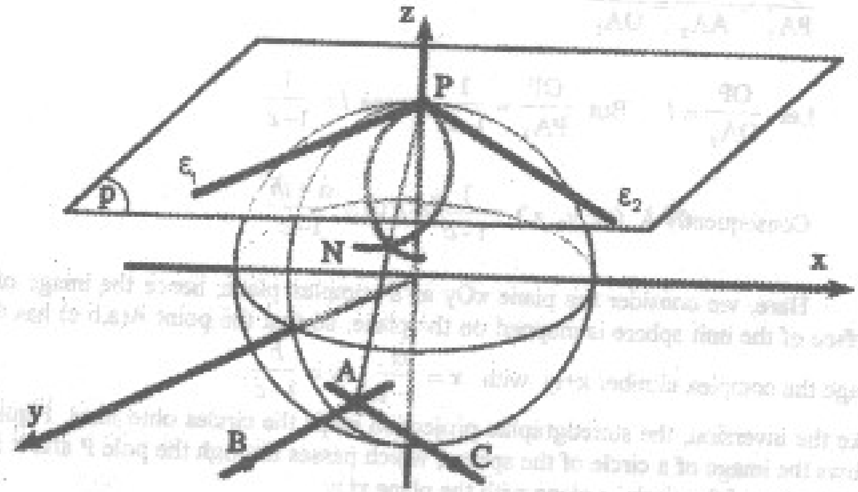


Figure 15

3.3. The infinite point of the complex plane

As we have seen the stereographic projection maps the surface of the unit sphere, apart from the pole, onto the complex plane.

In order to eliminate this exception we introduce a new point of the complex plane which corresponds to the pole P.

This point is called the **infinite point** of the complex plane and is symbolized by ∞.

This new point is not combined **algebraically** with the complex numbers, but as an infinite point can be defined **geometrically** as the limit of every sequence of points (complex numbers) whose modulus tend to the real infinite. On complex plane for $f(z) = \frac{1}{z}$ we have that $f(\infty) = 0$, $f(0) = \infty$.

By introducing the infinite point we succeed that the stereographic projection defines an one - to - one correspondence between the whole surface of the unit sphere and the complex plane including ∞.

Hence, every transformation in one of these two set of points is, to the other. For such transformations we will use the term lift. So, for example, if f(z) is a rotation on the complex plane having centre the origin O(o,o), then this is through the stereographic projection, that is through a lift to the surface of the unit sphere, a rotation about z - axis.

3.3.1 Definition

Let $S: A \rightarrow B$ be a continuous function and $S(A)=B$. In this case we say that S is a covering or a transformation covering from A to B or that A covers B through S. Also, let $f: B \rightarrow B$ be a transformation . A transformation $g: A \rightarrow A$ is called lift of f, if $S(g(z))= f(S(z))$ for every z. The above definitions are illustrated in the following diagram.

Ptolemeos was the first one who used the stereographic projection (2nd century A.C.) and with its help he constructed astronomical instruments like astrolabe, who counts the coordinates of the stars in the celestial sphere e.t.c. The basic properties of the stereographic projection appear presented in Ptolemeos' book "Planespherium"

4. The "Erlanger program"

As we mentioned, this program was presented for the first time at the University of Erlanger city in the course of the first lecture given by Felix Klein in 1872. In this lecture, later on known as "Erlanger's program " , F. Klein describes a Geometry by the help of its group of symmetries e.g. the Euclidean Geometry is characterized by its isometries , which preserve the distances and consequently are directly related to concepts such as length, equality (equivalence), angle' s measure, collinear points e.t.c.

In order to understand Klein' s ideas we have to understand the way he conceives the equality of geometrical figures e.t.c. According to Klein the equality is a function (a transformation) which is **reflexive, symmetric** and **transitive**. Furthermore the set M of transformations has the properties:

a) The identity transformation $f(z)=z$ belongs to M.
b) If $f \in M$, then f can be inversed and $f^{-1} \in M$.
c) If f, $g \in M$, then $f \circ g \in M$.

Furthermore, we define as a group of transformations, a set of transformations G which has the properties:

a) G contains the identity transformation
b) The transformations of G are invertible and their inverses belong to G.
c) G is closed, with respect to transformations' composition.

From the above, we can define that:

Geometry is an ordered pair (S,G) where S is a non empty set and G is a group of transformations $f: S \rightarrow S$.

G is a group of transformations in Geometry.

This way of defining Geometry initiated different ways of studying new abstract mathematical structures and gave birth to the rapid development of Geometry in 20th century.

Abstract

In this note the concept of the complex transformation and its importance in the study of Geometry is considered. The paper begins with the basic properties of complex numbers. The main part of the paper is devoted to the study of several complex transformations. The last part of the paper considers the infinite point of the complex plane, the concept of the covering transformation and the definition of Geometry with the help of transformations defined on the complex plane.

REFERENCES

1. Klein, F., Geometry, New York 1939.
2. Klein, F., The Icosohedron, Dover New York 1956
3. Martin, G., Tranformation Geometry, New York 1982.
4. Rosenfeld, B. A., Sergeeva, N. D., Stereographic Projection, Mir Moscow 1977.

National and Capodistrian University of Athens
33, Ippokrutus Str., 10680 Athens Greece

Primit la 1.11.1997