Lucrările Seminarului de
CREATIVITATE MATEMATICĂ
Vol. 9(1999-2000), 15 - 22

METODE DE DETERMINARE A UNOR PRIMITIVE CONDIŢIONATE

D.M.BĂTINEŢU- GIURGIU si Augustin SEMENESCU

In aceasta lucrare vom prezenta câteva propozitii care se constituie in tot atâtea metode unitare de abordare a unor clase de probleme publicate in diferite reviste sau culegeri de probleme de matematica.

Propozitia 1 . Fie $f, g : \mathbf{R} \to \mathbf{R}_+^*$ functii derivabile iar $u, v : \mathbf{R} \to \mathbf{R}$ functii continue cu proprietatile :

$$f'(x) = u(x) \cdot f(x),\ g'(x) = u(x) \cdot g(x) - v(x), \forall x \in \mathbf{R} \tag{1}$$

Daca exista numerele reale nenule a, b si $I \subset \mathbf{R}$ (I interval) astfel incât $a \cdot f(x) + b \cdot g(x) \neq 0,\ \forall x \in I$, atunci primitiva functiei $h : I \to \mathbf{R}$, $h(x) = \dfrac{f(x) \cdot v(x)}{(a \cdot f(x) + b \cdot g(x))^2}$ este functia

$$H : I \to \mathbf{R},\ H(x) = \frac{1}{b} \cdot \frac{f(x)}{a \cdot f(x) + b \cdot g(x)} \tag{2}$$

Demonstratie . Observam ca

$$H'(x) = \frac{1}{b} \cdot \left(\frac{f(x)}{a \cdot f(x) + b \cdot g(x)}\right)' =$$

$$= \frac{1}{b} \cdot \frac{f'(x) \cdot (a \cdot f(x) + b \cdot g(x)) - f(x) \cdot (a \cdot f'(x) + b \cdot g'(x))}{(a \cdot f(x) + b \cdot g(x))^2} =$$

$$= \frac{1}{b} \cdot \frac{b \cdot f'(x) \cdot g(x) - b \cdot f(x) \cdot g'(x)}{(a \cdot f(x) + b \cdot g(x))^2} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(a \cdot f(x) + b \cdot g(x))^2} =$$

$$= \frac{u(x) \cdot f(x) \cdot g(x) - f(x) \cdot g'(x)}{(a \cdot f(x) + b \cdot g(x))^2} = \frac{f(x) \cdot (u(x) \cdot g(x) - g'(x))}{(a \cdot f(x) + b \cdot g(x))^2} =$$

$$= \frac{f(x) \cdot (u(x) \cdot g(x) - u(x) \cdot g(x) + v(x))}{(a \cdot f(x) + b \cdot g(x))^2} = \frac{f(x) \cdot v(x)}{(a \cdot f(x) + b \cdot g(x))^2} = h(x) ,$$

ceea ce demonstreaza propozitia .

Propozitia 2 . Fie $f, g : \mathbf{R}_+ \to \mathbf{R}_+$ functii derivabile iar $u, v : \mathbf{R}_+ \to \mathbf{R}$ functii continue astfel incât

$$f'(x) = u(x) \cdot f(x), \quad g'(x) = u(x) \cdot g(x) - v(x), \forall x \in \mathbf{R}_+ \qquad (3)$$

Daca exista $a \in \mathbf{R}_+$, $b \in \mathbf{R}_+^*$ si intervalul $I \subset \mathbf{R}_+$ cu proprietatea ca $a \cdot f(x) + b \cdot g(x) > 0$, $\forall x \in I$, atunci primitiva functiei, $h : I \to \mathbf{R}$, $h(x) = \dfrac{v(x)}{a \cdot f(x) + b \cdot g(x)}$ este functia $H : I \to \mathbf{R}$, $h(x) = \dfrac{1}{b} \cdot U(x) - \dfrac{1}{b} \cdot \ln \ (a \cdot f(x) + b \cdot g(x))$ unde $U : \mathbf{R}_+ \to \mathbf{R}$ este o primitiva a functiei u .

Demonstratie . Intr-adevar ,

$$H'(x) = \frac{1}{b} \cdot U'(x) - \frac{1}{b} \cdot \frac{(a \cdot f(x) + b \cdot g(x))'}{a \cdot f(x) + b \cdot g(x)} =$$

$$= \frac{1}{b} \cdot u(x) - \frac{1}{b} \cdot \frac{a \cdot f'(x) + b \cdot g'(x)}{a \cdot f(x) + b \cdot g(x)} =$$

$$= \frac{a \cdot u(x) \cdot f(x) + b \cdot u(x) \cdot g(x) - a \cdot f'(x) - b \cdot g'(x)}{b \cdot (a \cdot f(x) + b \cdot g(x))} =$$

$$= \frac{a \cdot (u(x) \cdot f(x) - f'(x)) + b \cdot (u(x) \cdot g(x) - g'(x))}{b \cdot (a \cdot f(x) + b \cdot g(x))} =$$

$$= \frac{b \cdot v(x)}{b \cdot (a \cdot f(x) + b \cdot g(x))} = \frac{v(x)}{a \cdot f(x) + b \cdot g(x)} = h(x)$$

ceea ce demonstreaza propozitia .

In continuare vom considera polinomul $P_n = 1 + \frac{X}{1!} + \frac{X^2}{2!} + ... + \frac{X^n}{n!} \in$ $\in \mathbf{R}_+^*[X]$. Observam ca functia $g : \mathbf{R} \to \mathbf{R}$, $g(x) = P_n(x)$ verifica ecuatia diferentiala

$$y - y' = \frac{x^n}{n!} \tag{4}$$

iar functia $f : \mathbf{R} \to \mathbf{R}$, $f(x) = \mathrm{e}^x$ verifica ecuatia diferentiala omogena asociata ecuatiei (4) adica $y - y' = 0$.

Aplicatia 1.1. Sa se calculeze

$$\int \frac{x^n \cdot \mathbf{e}^x}{(a \cdot \mathrm{e}^x + P_n(x))^2}\,\mathrm{d}x, \quad \text{unde} \quad n \in \mathbf{N}^* \quad \text{si} \quad a \in \mathbf{R} \tag{5}$$

D. Acu, (19320), G.M. 7/1982, pag. 284

Solutie. In propozitia 1 luam $b = 1$, $a \in \mathbf{R}$, $I = \mathbf{R}$, $u(x) = 1$, $\forall x \in \mathbf{R}$, $f(x) = \mathrm{e}^x$, $g(x) = P_n(x)$, atunci $v(x) = \frac{x^n}{n!}$, $\forall x \in \mathbf{R}$ si deci

$$\int \frac{x^n \cdot \mathbf{e}^x}{(a \cdot \mathrm{e}^x + P_n(x))^2}\,\mathrm{d}x = \frac{n! \cdot \mathrm{e}^x}{a \cdot \mathrm{e}^x + P_n(x)} + \mathcal{C} \; .$$

Aplicatia 1.2. Sa se determine primitiva functiei

$$f : \mathbf{R}_+ \to \mathbf{R}_+^*,\ h(x) = \frac{\mathbf{e}^x \cdot (2 \cdot n! \cdot \sin x + x^n)}{(\mathrm{e}^x + P_n(x) + \sin x + \cos x)^2} \tag{6}$$

Solutie. In propozitia 1 luam $f(x) = \mathrm{e}^x$, $u(x) = 1, \forall x \in \mathbf{R}_+$, $g(x) =$ $= P_n(x) + \sin x + \cos x$, $a = b = 1$ si astfel obtinem

$$v(x) = g(x) - g'(x) = P_n(x) + \sin x + \cos x - P_{n-1}(x) - \cos x + \sin x = \frac{x^n}{n!} + 2\sin x \; .$$

Prin urmare

$$\int h(x) \cdot \mathrm{d}x = \int \frac{\mathbf{e}^x \cdot v(x)}{(\mathrm{e}^x + f(x))^2}\,\mathrm{d}x = \frac{\mathbf{e}^x}{\mathrm{e}^x + f(x)} + \mathcal{C} \Leftrightarrow$$

$$\Leftrightarrow \int \frac{\mathbf{e}^x\left(\frac{x^n}{n!}+2\sin x\right)}{(\mathbf{e}^x+f(x))^2}\,\mathrm{d}x = \frac{\mathbf{e}^x}{\mathbf{e}^x+f(x)}+\mathcal{C} \Leftrightarrow$$

$$\Leftrightarrow \int \frac{\mathbf{e}^x\cdot(2\cdot n!\cdot\sin x+x^n)}{(\mathbf{e}^x+f(x))^2}\,\mathrm{d}x = \frac{n!\cdot\mathbf{e}^x}{\mathbf{e}^x+f(x)}+\mathcal{C}\,.$$

Aplicatia 1.3. Fie $g:\mathbf{R}_+\to\mathbf{R}_+$ o functie derivabila si $v:\mathbf{R}_+\to\mathbf{R}$ o functie continua astfel inc'qt $g(x)=g'(x)+v(x)$, $\forall x\in\mathbf{R}_+$. Sa se determine

$$\int \frac{\mathbf{e}^x\cdot v(x)}{(\mathbf{e}^x+g(x))^2}\,\mathrm{d}x \tag{7}$$

Solutie. Deoarece v si g sunt functii continue pe $\mathbf{R}_+$ rezulta ca $g'=g-v$ este continua pe $\mathbf{R}_+$. Prin urmare in propozitia 1 luam $a=b=1$, $f(x)=$ $=\mathbf{e}^x$, $u(x)=1$, $\forall x\in\mathbf{R}_+$ si astfel $v(x)=g(x)-g'(x)$ de unde deducem ca

$$\int \frac{\mathbf{e}^x\cdot v(x)}{(\mathbf{e}^x+g(x))^2}\,\mathrm{d}x = \frac{\mathbf{e}^x}{\mathbf{e}^x+g(x)}+\mathcal{C}\,.$$

Aplicatia 2.1. Fie $f,g:\mathbf{R}_+\to\mathbf{R}_+$ cu f derivabila , g continua astfel incât $f(x)=g(x)+f'(x)$. Daca $a,b,c\in\mathbf{R}$, nu toate nule si $I\subset\mathbf{R}_+$ este un interval cu proprietatea ca $a\cdot\mathbf{e}^x+b\cdot f(x)+c\cdot P_n(x)\in\mathbf{R}^*$, $\forall x\in I$, atunci

$$\int \frac{b\cdot n!\cdot g(x)+c\cdot x^n}{a\cdot\mathbf{e}^x+b\cdot f(x)+c\cdot P_n(x)}\mathrm{d}x = n!\cdot(\mathbf{e}^x-\ln|a\cdot\mathbf{e}^x+b\cdot f(x)+c\cdot P_n(x)|)+\mathcal{C}\ ,(\forall).$$

D.M.Bătineţu-Giurgiu ([1]) .

Demonstratie. Se observa ca

$$\frac{b\cdot n!\cdot g(x)+c\cdot x^n}{a\cdot\mathbf{e}^x+b\cdot f(x)+c\cdot P_n(x)} = n!\cdot\frac{b\cdot g(x)+c\cdot\frac{x^n}{n!}}{a\cdot\mathbf{e}^x+b\cdot f(x)+c\cdot P_n(x)} =$$

$$= n!\cdot\frac{b\cdot(f(x)-f'(x))+c\cdot(P_n(x)-P_{n-1}(x))}{a\cdot\mathbf{e}^x+b\cdot f(x)+c\cdot P_n(x)} =$$

$$= n!\cdot\frac{a\cdot(\mathbf{e}^x-\mathbf{e}^x)+b\cdot(f(x)-f'(x))+c\cdot(P_n(x)-P_{n-1}(x))}{a\cdot\mathbf{e}^x+b\cdot f(x)+c\cdot P_n(x)} =$$

$$= n! \cdot \left(1 - \frac{(a \cdot \mathbf{e}^x + b \cdot f(x) + c \cdot P_n(x))'}{a \cdot \mathbf{e}^x + b \cdot f(x) + c \cdot P_n(x)}\right)$$

de unde rezulta relatia din enunt .

Aplicatia 2.2. Sa se calculeze

$$\int_0^1 \frac{x^n}{\mathbf{e}^x + P_n(x)} \, \mathrm{d}x \, .$$

Cristinel Mortici , (23514), G.M. 3/1996, pag. 181

Solutie . Conform propozitei 2 rezulta ca

$$\int \frac{x^n}{\mathbf{e}^x + P_n(x)} \, \mathrm{d}x = n! \cdot (x - \ln \ (\mathbf{e}^x + P_n(x))) + \mathcal{C}$$

si atunci

$$\int_0^1 \frac{x^n}{\mathbf{e}^x + P_n(x)} \, \mathrm{d}x = n! \cdot \left(x|_0^1 - \ln \ (\mathbf{e}^x + P_n(x))|_0^1\right) =$$

$$= n! \cdot \left(1 - \ln \ \frac{\mathbf{e} + P_n(1)}{2}\right) .$$

Aplicatia 2.3. Sa se calculeze $\int_0^1 \frac{f(x) - f'(x)}{\mathbf{e}^x + f(x)} \, \mathrm{d}x$ unde $f : \mathbf{R}_+ \to \mathbf{R}_+$ este o functie derivabila cu derivata continua .

D.M.Bătineţu-Giurgiu , (23875), G.M. 2/1998, pag. 85

Solutie. Este evident ca

$$\frac{f(x) - f'(x)}{\mathbf{e}^x + f(x)} = \frac{\mathbf{e}^x + f(x) - (\mathbf{e}^x + f(x))'}{\mathbf{e}^x + f(x)} = 1 - \frac{(\mathbf{e}^x + f(x))'}{\mathbf{e}^x + f(x)}$$

de unde rezulta ca

$$\int_0^1 \frac{f(x) - f'(x)}{\mathbf{e}^x + f(x)} \, \mathrm{d}x = (x - \ln \ (\mathbf{e}^x + f(x))|_0^1 = 1 - \ln \ \frac{\mathbf{e} + f(1)}{1 + f(0)} .$$

Aplicatia 2.4. Sa se calculeze

$$\int \frac{x - 1 + \sin x - \cos x}{\mathbf{e}^x + x + \sin x} \, \mathrm{d}x \, .$$

N. Sârbu si **Silviu Stossel** , Olimpiada judeteana Giurgiu 1990

Solutie. In propozitia 2 consideram $a = b = 1$, $f(x) = \mathbf{e}^x$, $g(x) = = x + \sin x$ si astfel obtinem

$$\int \frac{x - 1 + \sin x - \cos x}{\mathbf{e}^x + x + \sin x} \, dx = x - \ln\left(\mathbf{e}^x + x + \sin x\right) + \mathcal{C} \; .$$

Aplicatia 2.5. Sa se determine primitiva functiei urmatoare $f : \mathbf{R}_+ \to \mathbf{R}$, $f(x) = \dfrac{x^n + n! \cdot a^x \cdot (1 - \ln a)}{a^x + P_n(x)}$ unde $a \in (0,1) \cup (1, \infty)$, $n \in \mathbb{N}^*$ si unde $P_n = 1 + \dfrac{X}{1!} + \dfrac{X^2}{2!} + + \ldots + \dfrac{X^n}{n!}$.

Solutie. Fie $g : \mathbf{R}_+ \to \mathbf{R}$, $g(x) = a^x + P_n(x)$, functie derivabila cu $g'(x) = a^x \cdot \ln a + P_{n-1}(x)$

Rezulta atunci ca

$$g(x) - g'(x) = a^x \cdot (1 - \ln a) + \frac{x^n}{n!} \Leftrightarrow n! \cdot (g(x) - g'(x)) = n! \cdot a^x (1 - \ln a) + x^n \; .$$

Cu aceasta primitiva functiei din enunt este

$$\int f(x) \, dx = \int_{g'(x)} \frac{n! \cdot (g(x) - g'(x))}{g(x)} \, dx = n! \cdot \int dx -$$

$$-n! \cdot \int \frac{g'(x)}{g(x)} \cdot dx = n! \cdot x - n! \cdot \ln|g(x)| + \mathcal{C} =$$

$$= n! \cdot x - n! \cdot \ln g(x) + C = n! \cdot x - n! \cdot \ln \quad (a^x + P_n(x)) + \mathcal{C} \; .$$

In continuare recomandam cititorilor sa rezolve folosind metodele prezentate in propozitiile 1 si 2 urmatoarele probleme :

1. Sa se calculeze

$$\int \frac{a \cdot \cos x}{b \cdot \mathbf{e}^{\;x} + c \cdot (\sin x + \cos x)}) \cdot dx \quad .$$

Problema 21, a) din [3], pag.6.

2. Sa se calculeze

$$\int \frac{5 + \sin 2x + 3 \cdot \sin x + 5 \cdot \cos x}{\sin x + \cos x + 2} \cdot dx \quad .$$

C.Caragea, Olimpiada judeteana Constanta, 1991

3. Sa se determine primitiva functiei $f:(1,\infty)\to\mathbf{R}$,

$$f(x)=\frac{x+\sin^2 x+\sin 2x}{x-\cos^2 2x}\ .$$

Dan Seclăman , problema 1.115 din [2], pag. 11

4. Sa se determine primitiva functiei $f:\left(0,\dfrac{\pi}{2}\right)\to\mathbf{R}$,

$$f(x)=\frac{(x+1)\cdot\sin x+(4-x)\cdot\cos x+4}{(x+\sin x)\cdot(3+\sin x+\cos x)}\ .$$

Dan Seclăman, problema 1.116 din [2], pag.12.

Bibliografie

1. **Bătineţu-Giurgiu M.D.**, ASUPRA PROBLEMEI 2354 DIN G.M. 3/1996, Revista de Matematica din Craiova, CARDINAL Anul IX, Nr.1, 1998/1999, pag. 8

2. **Bătineţu-Giurgiu M.D., Bătineţu-Giurgiu Maria, Bîrchi-Damian Ion, Chiţimia Viorel**, PRIMITIVE SI INTEGRALE, Editura BÎIRCHI, Timisoara, 1998 .

3. **Lupu Tudorel**, PROBLEME DE ANALIZA MATEMATICA, Editura GIL, Zalau, 1996 .

A UNITARY WAY TO DETERMINE SOME INDEFINITE INTEGRALS

Abstract. In this paper the determination of the indefinite integrals for functions constructed by means of the solutions of some differential equations is approached in an unitary way. Numerous applications to the propozitions proved in this paper are given .

Primit: 18.10.2000

D.M. Bătineţu-Giurgiu
Colegiul Naţional Matei Basarab
Str. Matei Basarab 32,
Bucureşti

Augustin Semenescu
Universitatea Politehnică Bucureşti
Splaiul Independeţei 313
Sector 6, cod 77206