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ON KUMMER'S CONVERGENCE CRITERIA

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Positive terms series are an important part of the mathematical analysis. The study of the convergence of series of this type interests many mathematicians, in particular convergent series having many applications in various domains. In this paper we present some convergence criteria based on the Kummer convergence criterion.

Definition 1. A series $\sum_{n=1}^{\infty} a_n$ is called a positive terms series if a_n is a positive for every $n = 1, 2, ..., \infty$.

Theorem 1 (KUMMER). Let $a_1, a_2, ...a_n, ...,$be a sequence of positive numbers. If there is a fixed $k > 0$ such that

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} > k$$

for every n, then the series $\sum_{n=1}^{\infty} u_n$ is convergent.

If $a_n \frac{u_n}{u_{n+1}} - a_{n+1} \leq 0$ for every n and the series $\sum_{n=1}^{\infty} \frac{1}{u_n}$ is divergent then also the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Proof. In the first case we have

$$u_{n+1} < \frac{1}{k}(a_n u_n - a_{n+1} u_{n+1})$$

hence

$$u_2 < \frac{1}{k}(a_1 u_1 - a_2 u_2)$$

....

$$u_{n+1} < \frac{1}{k}(a_n u_n - a_{n+1} u_{n+1})$$

and

$$S_{n+1} = u_1 + u_2 + ... + u_{n+1} < \frac{1}{k}(a_1 u_1 - a_{n+1} u_{n+1}) + u_1 < u_1 + \frac{1}{k} a_1 u_1.$$

The sequence S_n is convergent since it is increasing and upper bounded. In the second case we have

$$a_n u_n - a_{n+1} u_{n+1} \leq 0$$

or

$$\frac{a_n}{a_{n+1}} \leq \frac{u_{n+1}}{u_n} \Leftrightarrow \frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} \leq \frac{u_{n+1}}{u_n}$$

Using the comparison criterion and the divergence of series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ we obtain that the series $\sum_{n=1}^{\infty} u_n$ is divergent as well.

Remark. 1). In application one calculates

$$\lim_{n \to \infty} \left(a_n \frac{u_n}{u_{n+1}} - a_{n+1} \right) = \lambda$$

a) If $\lambda > 0$ the series $\sum_{n=1}^{\infty} u_n$ is convergent.
b) If $\lambda < 0$ the series $\sum_{n=1}^{\infty} u_n$ is divergent.

2) If $a_n = 1$we obtain the d' Alembert (rapport) criterion.

3) If $a_n = n$ we obtain the Duhamel criterion.

Corollary 1. Let us consider the sequence $a_n = n^p$ where $p \in [0, 1]$ and $\sum_{n=1}^{\infty} u_n$ a positive terms series.

If there is a fixed $k > 0$ such that

$$n^p \frac{u_n}{u_{n+1}} - (n+1)^p > k$$

then the series $\sum_{n=1}^{\infty} u_n$ is convergent.

If

$$n^p \frac{u_n}{u_{n+1}} - (n+1)^p \leq 0$$

then the series $\sum_{n=1}^{\infty} u_n$ is divergent.

The proof of this corollary is based on the proof of the previous theorem with the sequence a_n chosen to be $a_n = n^p$.

Remark.1) If $p = 0$ we obtain the d'Alembert (rapport) criterion.

2) If $p = 1$ we obtain the Duhamel criterion. We note that this corollary is a more general form of Raabe-Duhamel criterion which can be obtained from if we set $p = 1$.

Remark. In application one evaluates

$$\lim_{n\to\infty}\left(n^p\frac{u_n}{u_{n+1}}-(n+1)^p\right)=k$$

a)If $k>0$ the series $\sum_{n=1}^{\infty}u_n$ is convergent.
b) If $k<0$ the series $\sum_{n=1}^{\infty}u_n$ is divergent.
Corollary 2. Let us consider the sequence $a_n=\ln n,\ n>0$ such that $a_n>0$.
If $\sum_{n=1}^{\infty}u_n$ is a positive terms series and there is a $k>0$ such that:
(i) $\ln n\left(\frac{u_n}{u_{n+1}}\right)-\ln(n+1)>k$ then the series $\sum_{n=1}^{\infty}u_n$ is convergent
(ii) $\frac{u_n}{u_{n+1}}\le\frac{\ln(n+1)}{\ln n}$ then the series $\sum_{n=1}^{\infty}u_n$ is divergent.
Remark. If we evaluate

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}\le\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}=1$$

we obtain the divergence condition from the rapport (D'Alembert) criterion.
Corollary 3. Let us consider the sequence $a_n=\frac{1}{n^2}, a_n>0,\forall n\in\mathbf{N}^*$.
If $\sum_{n=1}^{\infty}u_n$ is a positive terms series and there is a fixed $k>0$ such that:
(i) If

$$\frac{1}{n^2}\frac{u_n}{u_{n+1}}-\frac{1}{(n+1)^2}>k,$$

then the series $\sum_{n=1}^{\infty}u_n$ is convergent.
(ii) If

$$\frac{u_n}{u_{n+1}}\le\frac{n^2}{(n+1)^2},\forall n\in\mathbf{N}^*,$$

then the series $\sum_{n=1}^{\infty}u_n$ is divergent.
Corollary 4. Let us consider the sequence

$$a_n=\frac{1}{n!},a_n>0,\forall n\in\mathbf{N}^*$$

If $\sum_{n=1}^{\infty}u_n$ is a positive terms series and there is a fixed $k>0$ such that:
(i) If

$$\frac{1}{n!}\frac{u_n}{u_{n+1}}-\frac{1}{(n+1)!}>k$$

then the series $\sum_{n=1}^{\infty}u_n$ is convergent.
(ii) If

$$\frac{u_n}{u_{n+1}}\le\frac{1}{n+1},\forall n\in\mathbf{N}^*$$

then the series $\sum_{n=1}^{\infty} u_n$ is divergent.

These convergence criteria can be used to investigate the convergence of some positive terms series for which other classical convergence criteria cannot be applied. Because they are based on the Kummer criterion, they can also be called Kummer-type criteria.

References

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Abstract. Positive terms series are an important part of the mathematical analysis. The study of the convergence of series of this type interests many mathematicians, in particular convergent series having many applications in various domains. In this paper we present some convergence criteria based on the Kummer convergence criterion.

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