Lucrările Seminarului de
CREATIVITATE MATEMATICĂ
Vol. 9(1999 - 2000), 81 - 90

COMPUTER FACILITATION IN UNDERSTANDING SOME NOTIONS IN ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction

Differential equations are one of basic area of higher mathematics. Their wide application in description of phenomena occurring in the real world makes they also form an essential part of physics, and the world can not be recognized without these equations [5]. Thus there is very important to dominate and deeply understand their nature, to feel the idea itself of the differential equation. It relates directly to the direction field. The importance of this concept is justified not only by the fact that almost all books on differential equations (e.g. [2]-[4]) discuss it. In some cases there is the direction filed which lets to find out a rough graphical solution to problems which can not be solved analytically (and numerical solutions are hard to be produced).

We deal with any function f defined in an area D contained in the real plane R_2. equipped with the rectangular co-ordinate system Oxy. We distinguish a set D_p of points in D. In most cases they are nodes of regular rectangular net, so the index p is a pair of two numbers, m and n, saying how many different abscissas x_j and different ordinates y_k build the set D_p.

Then we write $D_p = D_{m,n} := \{ (x_j, y_k): j=0..m, k=0..n\}$.

A ***direction field*** of a differential equation

(1) $$y' = f(x, y)$$

(generated by the set $D_{m,n}$, named often as a *canvas*) is the set

(2) $$\{ f(x_j, y_k) : (x_j, y_k) \in D_{m,n} \}.$$

A direction field may be (and usually it is) interpreted geometrically. To each point $(x_j, y_k) \in D_{m,n}$ we may assign the angle $\alpha_{j,k}$ such that

(3) $$\tan(\alpha_{j,k}) = f(x_j, y_k)$$

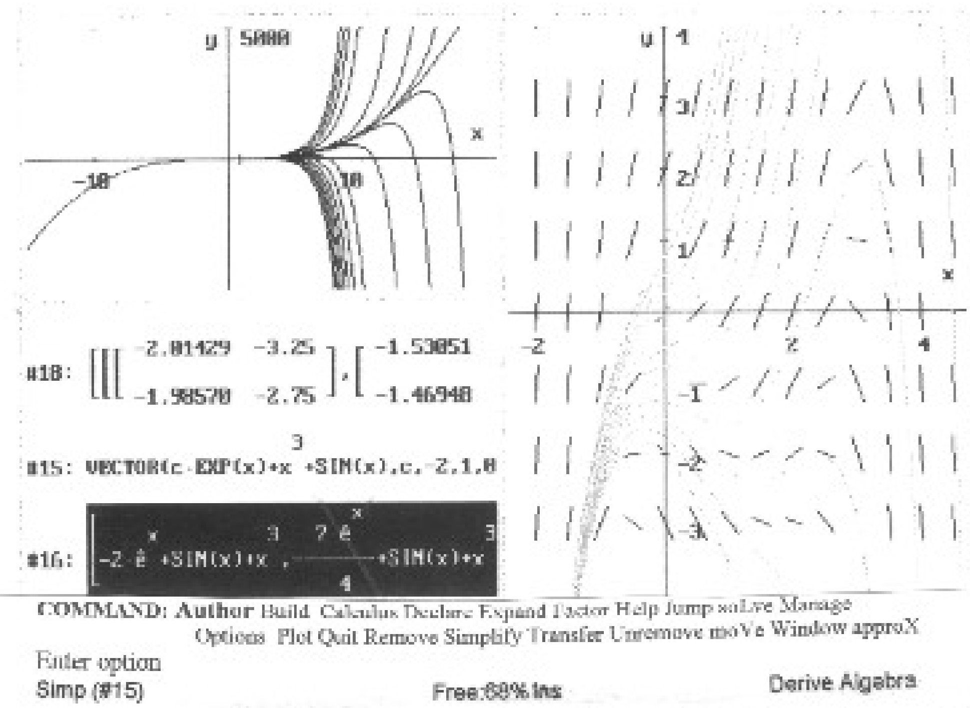

Fig.1. Direction field for the equation $y'(x) = y(x) + 3x^2 - x^3 + \cos(x) - \sin(x)$and some representatives ($c = -2, -1.75, -1.5, \ldots, 0.75$), 1 in Window 3, as well as in Window 2 where also there are plotted graphs for $c = -1/40, -1/400, -1/4000, 1/4000, 1/400$ and $1/400$) of the general solution $y(x) = c\cdot\exp(x) + x^3 + \sin(x)$ to this equation

and then we can draw a linear segment (called a ***direction***) sloped with this angle. Every such a segment indicates a direction along which a solution y of the equation $y' = f(x, y)$ locally goes. In other words, every solution y to the equation $y' = f(x, y)$ fits to these directions. Immediately from the form of considered equation we see that this fitting is of a tangent nature: a linear segment visualizing the direction lays on the tangent line to the curve $y = y(x)$. When enough linear segments are drawn (i.e. the net $D_{m,n}$ is dense enough), one can often see trends in the solution curves, thus one can plot a (rather rough than exact) graph of the curve described by the equation $F(x, y) = 0$ involving the independent variable x and the solution $y = y(x)$ of considered equation. In general, both the calculation of values $f(x_j, y_k)$ and the marking of the directions (i.e. linear segments) consume a lot of time. Modern computer technology can be applied here in aim to free a man from tedious (and mechanical) work. In this paper we share our experiments on how the direction fields (determined by ordinary differential equations of first order) can be produced with the help of computer. From a waste offer we take here the use of the computer algebra system DERIVE (version 3 released in 1995 [1]) from Soft Warehouse Inc. and a graphing program WinPlot (version compiled on 8^{th} of July 1999 [6]) written by Richard Parris working for Phillips Exeter Academy in Exeter (New Hampshire, USA). He generously allows free copying and distribution of the software, and provides frequent updates. The latest version can be downloaded from the website [6].

2. Tools provided in DERIVE and WinPlot

The computer algebra system DERIVE offers a wide collection of functions dealing with ordinary differential equations. In particular, the unit ODE_APPR.MTH stores the function **DIRECTION_FIELD** producing a matrix, the image of which is the direction field. An user gets this matrix by issuing the command **approX** on the call

DIRECTION_FIELD(f,x,x0,xm,m,y,y0,yn,n)

where

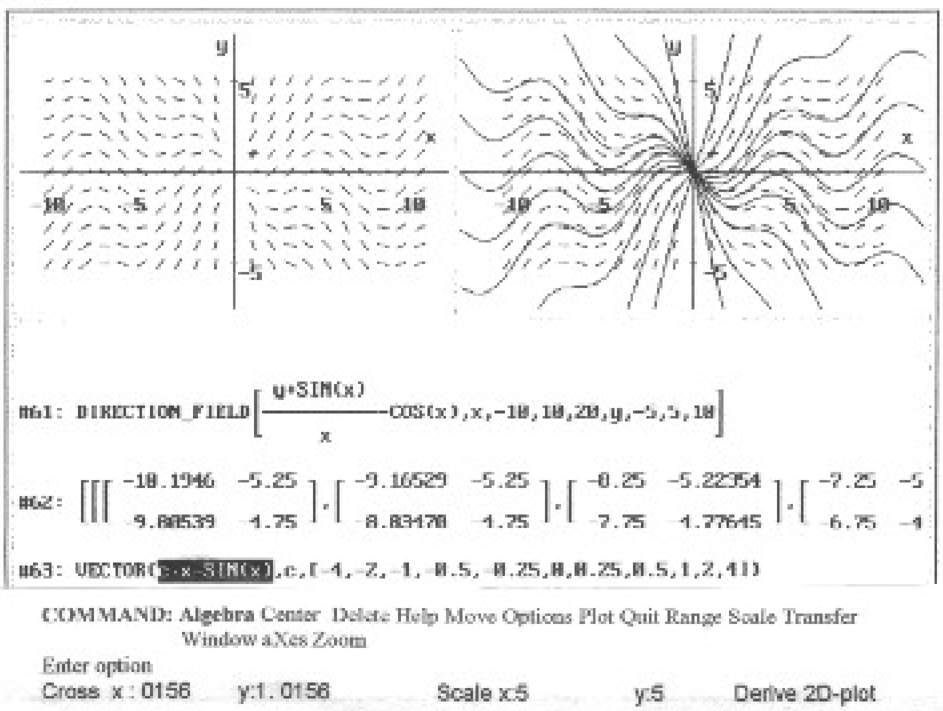

Fig.2. Direction field of the equation y' = {y+sin(x)}/x – cos(x) and graphs of eleven particular solutions of this equation

f stands for the right side of the equation (1),

x and **y** are names of co-ordinates constituting the rectangular system Oxy (**x** is called *abscissa* and is the horizontal co-ordinate, y is called *ordinate* and is the vertical one),

x0, **xm**, **m**, **y0**, **yn**, **n** determine the net $D_{m,n}$ of nodes

$$(x_j = \mathbf{x0}+j\cdot(\mathbf{xm}-\mathbf{x0})/\mathbf{m},\ y_k = \mathbf{y0}+k\cdot(\mathbf{yn}-\mathbf{y0})/\mathbf{n})$$

stating the canvas of the direction field (3). A direction (3) is the segment centered with respect to the abscissa x_j. If it is not perpendicular to the horizontal axis, it is spread over the interval of the constant length equal to $(\mathbf{xm}-\mathbf{x0})/(2\mathbf{m})$. The length of the segment measured along the vertical axis depends on the value.

The matrix produced by the the above approximation is next plotted (by issuing the command **Plot** while the mode **Connect** is on). To produce a direction filed in the program WinPlot we execute following steps:

a) choose the option **<u>W</u>indow <u>D</u>iff 2-dim** from the main menu of the program,

b) choose the option **<u>E</u>qua <u>D</u>iff eqn**; it appears the window **2-dim example** (see Fig.3),

c) edit (on the fields **x'** and **y'** accessible in the window **2-dim example**) expressions defining a new system of differential equations; we deal with the differential equation of the form (3), so we always insert 1 in the field **x'**, and f(x,y) in the field **y'**,

d) mark the radio button **vector field** (as it is shown in Fig.3) and activate the button **ok**.

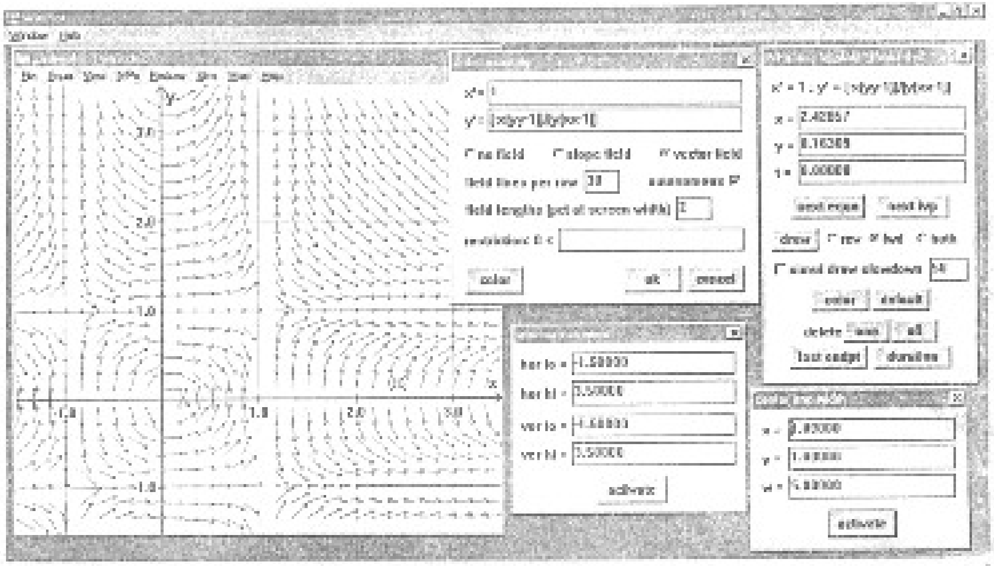

Fig.3. Direction field of the equation $y' = -\frac{x \cdot (1-y^2)}{y \cdot (1-x^2)}$ in the rectangle <–1.5, 3.5>×<–1.5, 3.5> and four decision boxes in the program WinPlot

Now it is drawn a direction field over the rectangle area determined by the values formed on fields **hor lo**, **hor hi** (lowest and highest values of the horizontal variable x, resp.), **ver lo** and **ver hi** (lowest and highest values of the variable y, respectively). These four editing fields are made accessible by the command **View Corners**, it namely open the window **window extremes**. This window is shown in Fig.3. Here it is also shown the window **center and width** which serves to define the visualisation area centered at indicated point (its co-ordinates are to be stated on the fields **x** and **y**).

The direction field is drawn on the canvas of nodes which are regularly distributed over all the windowed region. A direction has the form of an arrow a constant length. An user preferring to have no arrows, but only the slopes, has to switch the button **slope field** on. Then the option **vector field** is automatically turned off. Although now the visualisation takes view as made in DERIVE, both images differs because in DERIVE the slopes (i.e. directions) are centered with respect to the abscissa of the node and in WinPlot they start at the node (x_j, y_k).

3. Additional advantages

In DERIVE one can overlay graphs of particular solutions of the considered equation (1). For example, to plot five representatives of the family { $c\cdot\sin(x^2)$: $c \in R$ } resolving the equation $y' = 2x\cdot ctg(x^2)\cdot y$ we form the expression

VECTOR(c*SIN(x^2),c,-2,8,2)

next we simplify it and, finally, we issue command **Plot**.

We proceed analogously WinPlot. First we activate command **Equa Overlay** of the main menu, next we choose the option **y=f(x)...**. because we deal with explicit dependency y on x in the rectangular system Oxy (three more options offered here, namely that identified as **r=f(t)...**, **x=f(t)...** and **0=f(x,y)....**, but we do not choose they correspond to polar co-ordinates, parametric representations and implicit equations, respectively). The editing field is then appered and we edit a particular solution. It is immediately drawn after the button **ok** is activated.

WinPlot offers an additional possibility (which is not offered by DERIVE). An user can position the mouse cursor at any point (x_0, y_0) and click on the left button. If the option **Btns Trajectory LB** is on, it makes that there is drawn the graph of the particular solution passing through just selected point.

4. Four examples

In this part we present four examples which are found very instructive. Two first of them were elaborated by students themselves (during laboratory classes in the Faculty of Technical Physics at Poznań University of Technology), last two ones are taken from [2]. In this survey we take into account differential equations of various types. The criterion was about the regularity of solutions, singular points and singular solutions.

Example 1. Linear equation $y' = y+3x^2-x^3+\cos(x)-\sin(x)$. It has no singular points. It has regular solutions only, they are given by the formula $y = c\cdot\exp(x)+x^3+\sin(x)$, where c stands for an arbitrary number. In DERIVE 3.10 this solution can be produced by the simplification of the following call

LINEAR1_GEN(–1,3x^2–x^3+COS(x)–SIN(x),x,y)

to the function **LINEAR1_GEN** memorized in the utility file ODE1.MTH.

In Fig.1. there is shown the direction field. Directions, which compose this filed, are short linear segments in the window no.3.

They are the images (displayed in result of issuing the command **Plot**) of the matrix (having 14 rows and 7 columns) produced by the approximation of the call

DIRECTION_FIELD(y+3x^2–x^3+COS(x)–SIN(x),x,–,4.5,13,y,–3,3,6).

In the same window one can see 13 curves. They are solutions specified for the values –2 (the lowest curve), –1.75, –1.5, ..., 0.75, 1 (the curved traced above all other ones) of the parameter **c**. These solutions can be automatically generated in result of the simplification of the expression

VECTOR(c·EXP(x)+SIN(x)+x^3,c,–2,1,0.25).

Some other solutions are plotted in the window no.1. Here the range is much wider than that in the window no.3, so one can much easier note the shape of functions constituting the family $\{y = c \cdot e^x + \sin(x) + x^3 : x \in R\}$ of all solutions to investigated equation. Two representatives with slightly different values of c are almost identical for negative x's (the term x^3 dominate both others, so the graph of any particular solution coincides almost perfectly with the third-degree algebraic curve $y = x^3$) and become still more and more different as x approaches the infinity (for positive x's particular solutions seem to grow from the graph of the curve $y = x^3$).

Example 2. Linear equation $y' = \{y+\sin(x)\}/x - \cos(x)$. The general solution of this equation is the family $y = c \cdot x - \sin(x)$. The equation at hand has exactly one singular point, it is the origin (0,0) of the co-ordinate system Oxy. All solutions $y = y(x)$ of our equation satisfy the equation $y(0) = 0$, all integral curves passes through this point.

In Fig.2. one can see the direction field of considered equation plotted over the net of 21×11 nodes and graphs of 11 particular solutions of this equation.

Examples of equations having the infinitely many singular points are $y' = 2x/\tan(x^2) \cdot y$ and $y' = \sin(x)/\sin(y)$. Every one of these equations has infinitely many singular points. Examples of equations having finite numbers of singular points are given below.

Example 3. Equation $y' = -x/y \cdot (1-y^2)/(1-x^2)$. The origin (0,0) and all four vertices (−1,−1), (−1,1), (1,1), (1,−1) of the square centered at the origin are singular points of this equation. There exists no singular solution. The direction field is reproduced in Fig.3.

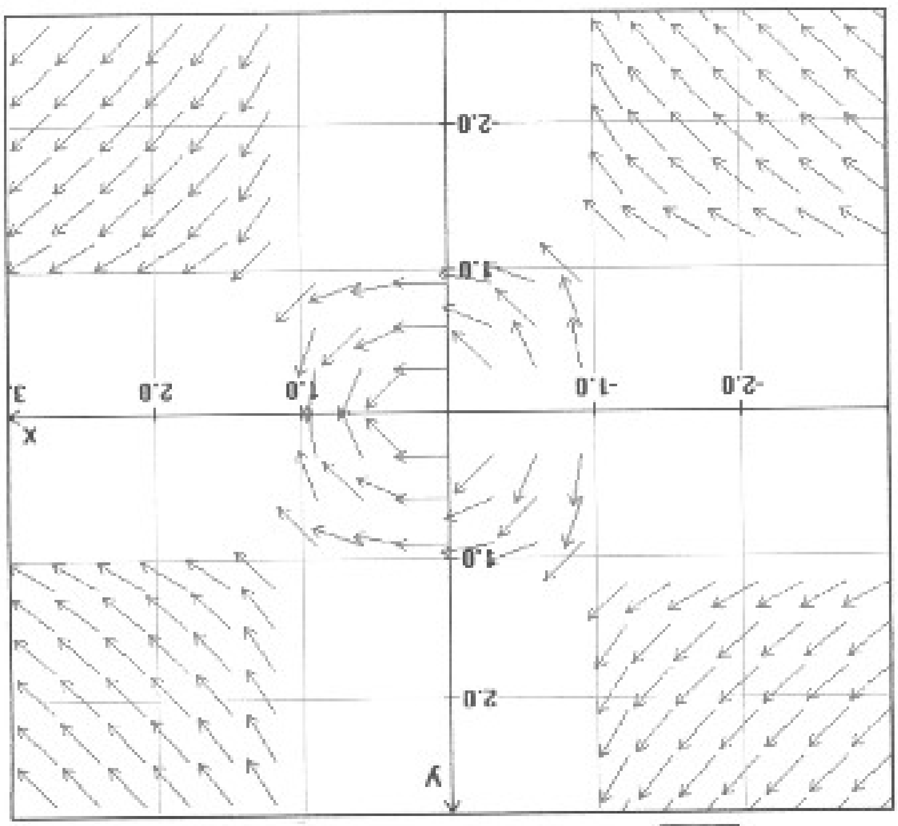

Fig.4. Direction field of the equation $\frac{dy}{dx} = \frac{-x\sqrt{1-y^2}}{y\sqrt{1-x^2}}$

Example 4. Equation $y' = -\frac{x}{y}\sqrt{(1-y^2)/(1-x^2)}$. In Fig.4 we see the direction field of this equation. Note that there are some regions where this equation determines no directions. Particular integrals exist in regions where $(1-x^2)/(1-y^2) > 0$. Here we have five singular points (they are the same points as in Example 3) and four singular solutions $y = -1$, $y=1$, $x=-1$, $x=1$.

5. Investigating the sensibility

In applications we often have equations of the form (1) which can be not integrated exactly. Then we apply numerical methods. They are realized in float-point arithmetic. In some cases (called "stiff ones") even small inevitable round-off errors can then dramatically affect results. The same concerns the sensibility of the solution: even small changes at a single point may essentially influence the output. A typical situation is for Cauchy problem, when instead of the exact value y_0 of searched solution y at point x_0 we have to take into account a perturbation ε. It means that we dealt with two initial problems:

$$\text{I. } y' = f(x, y) \quad y(x_0) = y_0. \qquad \text{II. } y' = f(x, y) \quad y(x_0) = z_0,$$

where $z_0 = y_0 + \varepsilon$.

In Fig.1 one can see graphs of some particular solutions. Its shape, and especially the condensation of curves in the neighborhood of the point $x_0 = -1$ says that the problem is very sensitive for the change of initial value y_0. Quite different situation takes place in Fig.3. Here, for instance, even a big perturbation at $x_0 = 1.3$ does not impact (or, more precisely, impacts in a very low degree) values at, let's say, $x = 3.5$ (all integral curves converge to one of asymptotic lines: $y = -1$ or $y = 1$).

But what to do if (just like in most cases describing the real world) we do not know the formula for the general integral of the equation at hand? Sometimes it is nothing else than to deduce the solution and its sensibility only on the base of direction field. The same field can be also used to verify the degree of reliability of the solution(s) provided by numerical method(s). If the graph sketched on the base of numerical procedure does not tangent-wise cling to directions forming the direction field, we are justified to say that the numerical treatment of considered equation gave an unsatisfactory result.

6. Conclusions

Differential equations are essential (and, in the same time, one of the most difficult) part of higher mathematics and physics. It is impossible to familiarize students to these equations if basic notions are not well explicated and deeply understood. The nature of the subject makes that getting instructive examples is time-consuming. In particular, it concerns the notions of a direction field and the sensibility to the initial conditions. In the paper, where we deal with equations of the form $y' = f(x,y)$ involving the independent variable x (related to the rectangular co-ordinate system Oxy) and the function y depending on x, it is shown how modern computer programs, namely DERIVE from Soft Warehouse Inc. and WinPlot created by Prof. Richard Parris, may facilitate the teacher's and student's work in this area.

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COMPUTER FACILITATION IN UNDERSTANDING SOME NOTIONS IN ORDINARY DIFFERENTIAL EQUATIONS

Key words: ordinary differential equations, computers and education

Abstract. Modern technology may facilitate a deep understanding notions which are basic for future progress in learning and professional activity. In particular, it concerns such advanced areas as differential equations, where in many cases one needs a lot of time to produce instructive examples explicating ideas such as a direction field and the sensibility to initial conditions. In the paper there is reported the effective and attractive use of two computer programs (DERIVE and WinPlot) in understanding these two notions in differential equations of the form y'=f(x,y) stating the relation between variables x and y=y(x) related to the rectangular co-ordinate system Oxy.

Primit: 16.10.2000

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