Lucrările Seminarului de
CREATIVITATE MATEMATICĂ
Vol. 11(2002), 43 - 49

ON THE IMPLICIT FUNCTION THEOREM

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1 Introduction

The implicit function theorem is a basic theorem of mathematical analysis and it has a major importance in many mathematical problems. This theorem has many formulations, see [5]. Every category of functions has its own special version of implicit function theorem and there are particular versions adapted to Banach spaces, algebraic geometry and to functions that are not even smooth.

The aim of this article is to state and to proof a version of analytic implicit function theorem for functions of two variables.

First, we need a result from the theory of functions of several complex variables [6], which allows us to bound the coefficients of a convergent power series.

Lemma 1.1 *If the function* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *has the power series development*

$$f(x, y, z) = \sum_{i,j,k=0}^{\infty} \gamma_{ijk} x^i y^j z^k , \quad \gamma_{ijk} \in \mathbb{R},\ \forall\, i, j, k \in \{0, 1, 2, \dots\}, \qquad (1.1)$$

absolutely convergent for $|x| \le r_1$, $|y| \le r_2$, $|z| \le r_3$ *and if*

$$|f(x, y, z)| \le M,\ \forall\, (x, y, z) \in \mathbb{R}^3 \text{ with } |x| \le r_1,\ |y| \le r_2,\ |z| \le r_3 \qquad (1.2)$$

then

$$|\gamma_{ijk}| \le \frac{M}{r_1^i r_2^j r_3^k}\,, \quad \forall\, i,j,k \in \{0,1,2,\dots\} \tag{1.3}$$

Let now, the equation

$$F(x,y,z) = 0\,, \tag{1.4}$$

be, where $F:\mathbb{R}^3 \longrightarrow \mathbb{R}$. Suppose that F has the power series development

$$F(x,y,z) = \sum_{i,j,k=0}^{\infty} a_{ijk}\, x^i y^j z^k\,, \tag{1.5}$$

absolutely convergent for $|x| \le r_1$, $|y| \le r_2$, $|z| \le r_3$.

2 Analytic Implicit Function Theorem

Now we can state for equation (1.4) an implicit function theorem, called analytic implicit function theorem.

Teorema 2.1 *Suppose that power series* (1.5) *is absolutely convergent for* $|x| \le r_1$, $|y| \le r_2$, $|z| \le r_3$. *If* $a_{000} = 0$ *and* $a_{001} \ne 0$, *then there exists* $r > 0$ *and the analytic implicit function* $f : \overline{B}_r(0) \longrightarrow \mathbb{R}$, *where* $\overline{B}_r(\mathbf{0})$ *is the ball from* $\mathbb{R}^2$ *with center* $\mathbf{0} = (0,0)$ *and radius* r, *such that*

$$F(x,y,f(x,y)) = 0\,, \quad \forall\,(x,y) \in \overline{B}_r(0)\,, \tag{2.1}$$

and the power series

$$f(x,y) = \sum_{\substack{i,j=0\\ i+j>0}}^{\infty} c_{ij}\, x^i y^j\,, \tag{2.2}$$

is absolutely convergent for $(x,y) \in \overline{B}_r(\mathbf{0})$.

Proof. The proof follows the line of the proof of theorem 2.4.4 from [5]. It will be no loss of generality to assume $a_{001} = 1$. Then the series (1.5) takes the form

$$F(x,y,z) = z + \sum_{i,j=1}^{\infty} (a_{i,j,0} + a_{i,j,1} z)\, x^i y^j + \sum_{i,j=0}^{\infty} \sum_{k=2}^{\infty} a_{ijk} x^i y^j z^k \tag{2.3}$$

If we put $b_{ijk} = -a_{ijk}$, $\forall\, i,j,k \in \{0,1,2,\dots\}$, the equation (1.4) can be rewrite as

$$z = \sum_{i,j=1}^{\infty} (b_{i,j,0} + b_{i,j,1} z)\, x^i y^j + \sum_{i,j=0}^{\infty} \sum_{k=2}^{\infty} b_{ijk}\, x^i y^j z^k\,, \tag{2.4}$$

or

$$z = B(x, y, z)\,, \tag{2.5}$$

where

$$B(x, y, z) := \sum_{i,j=1}^{\infty} (b_{i,j,0} + b_{i,j,1} z)\, x^i y^j + \sum_{i,j=0}^{\infty} \sum_{k=2}^{\infty} b_{ijk}\, x^i y^j z^k \tag{2.6}$$

Substituting $z = f(x, y)$ into (2.4) with $f(x, y)$ given by (2.2), we obtain

$$\begin{aligned} \sum_{\substack{i,j=0 \\ i+j>0}}^{\infty} c_{ij} x^i y^j &= \sum_{\substack{i,j=0 \\ i+j>0}}^{\infty} b_{i,j,0}\, x^i y^j + \sum_{\substack{i,j=0 \\ i+j>0}}^{\infty} \sum_{\substack{p,q=0 \\ p+q>0}}^{\infty} b_{i,j,1}\, c_{pq} x^{i+p} y^{j+q} + \\ &+ \sum_{i,j=0}^{\infty} \sum_{k=2}^{\infty} b_{ijk}\, x^i y^j \left(\sum_{\substack{p,q=0 \\ p+q>0}}^{\infty} c_{pq}\, x^p y^q \right)^k . \end{aligned} \tag{2.7}$$

If all the series in (2.7) are absolutely convergent then the order of summation can be rearranged and we can equate like powers of x and y on the left – hand and right – hand sides of (2.7). We obtain

$$c_{1,0} = b_{1,0,0}\,, \tag{2.8}$$

$$c_{0,1} = b_{0,1,0}\,, \tag{2.9}$$

$$c_{2,0} = b_{2,0,0} + b_{1,0,1}\, c_{1,0}\,, \tag{2.10}$$

$$c_{1,1} = b_{1,1,0} + b_{1,0,1} c_{0,1} + b_{0,1,1} c_{1,0}\,, \tag{2.11}$$

$$c_{0,2} = b_{0,2,0} + b_{0,1,1} c_{0,1} \tag{2.12}$$

$$c_{3,0} = b_{3,0,0} + b_{1,0,1}\, c_{2,0} + b_{2,0,1}\, c_{1,0} + b_{1,0,2}\, c_{1,0}^2 + 2b_{0,0,2}\, c_{1,0}\, c_{2,0}\,, \tag{2.13}$$

$$\begin{aligned} c_{2,1} &= b_{2,1,0} + b_{1,0,1}\, c_{1,1} + b_{0,1,1}\, c_{2,0} + b_{2,0,1}\, c_{0,1} + b_{1,1,1}\, c_{1,0} + \\ &+ b_{0,1,2}\, c_{1,0} + 2b_{1,0,2}\, c_{1,0}\, c_{0,1}\,, \end{aligned} \tag{2.14}$$

$$\begin{aligned} c_{1,2} &= b_{1,2,0} + b_{1,0,1}\, c_{0,2} + b_{0,1,1}\, c_{1,1} + b_{1,1,1}\, c_{0,1} + b_{0,2,1}\, c_{1,0} + \\ &+ b_{1,0,2}\, c_{0,1}^2 + 2b_{0,1,2}\, c_{1,0} \cdot c_{0,1}\,; \end{aligned} \tag{2.15}$$

$$\begin{aligned} c_{0,3} &= b_{0,3,0} + b_{0,1,1}\, c_{0,2} + b_{0,2,1}\, c_{0,1} + b_{0,1,2}\, c_{0,1}^2 + \\ &+ 2b_{0,0,2}\, c_{0,1} \cdot c_{0,2}\,; \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
c_{4,0} &= b_{4,0,0} + b_{1,0,1}\, c_{3,0} + b_{2,0,1}\, c_{2,0} + b_{3,0,1}\, c_{1,0} + b_{2,0,2}\, c_{1,0}^2 + \\
&+ 2b_{1,0,2}\, c_{1,0}\, c_{2,0} + b_{1,0,2}\, c_{1,0}^3\,, \qquad (2.17)
\end{aligned}
$$

$$
\begin{aligned}
c_{3,1} &= b_{3,1,0} + b_{1,0,1}\, c_{2,1} + b_{0,1,1}\, c_{3,0} + b_{2,0,1}\, c_{1,1} + b_{1,1,1}\, c_{2,0} + \\
&+ b_{1,1,2}\, c_{1,0}^2 + 2b_{1,0,2}\, c_{1,0}\, c_{1,1} + 2b_{0,1,2}\, c_{1,0}\, c_{2,0} + \\
&+ 2b_{2,0,2}\, c_{1,0} \cdot c_{0,1} + b_{0,1,2}\, c_{1,0}^3 + 3b_{1,0,2}\, c_{1,0}^2 \cdot c_{0,1}\,, \qquad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
c_{2,2} &= b_{2,2,0} + b_{1,0,1}\, c_{1,2} + b_{2,0,1}\, c_{0,2} + b_{1,1,1}\, c_{1,1} + b_{0,2,1}\, c_{2,0} + \\
&+ b_{2,0,2}\, c_{0,1}^2 + b_{0,2,2}\, c_{1,0}^2 + 2b_{1,0,2}\, c_{1,0}\, c_{0,2} + 2b_{1,0,2}\, c_{0,1}\, c_{1,1} + \\
&+ 2b_{0,1,2}\, c_{0,1}\, c_{2,0} + 2b_{0,1,2}\, c_{1,0}\, c_{1,1} + 2b_{1,1,2}\, c_{1,0}\, c_{0,1} + \\
&+ 3b_{1,0,2}\, c_{1,0}\, c_{0,1}^2 + 3b_{0,1,2}\, c_{1,0}^2\, c_{0,1}\,, \qquad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
c_{1,3} &= b_{1,3,0} + b_{1,0,1}\, c_{0,3} + b_{0,1,1}\, c_{1,2} + b_{1,1,1}\, c_{0,2} + b_{0,2,1}\, c_{1,1} + \\
&+ b_{1,1,2}\, c_{0,1}^2 + 2b_{1,0,2}\, c_{0,1}\, c_{0,2} + 2b_{0,1,2}\, c_{1,0}\, c_{0,2} + 2b_{0,2,2}\, c_{1,0}\, c_{0,1} + \\
&+ b_{1,0,2}\, c_{0,1}^3 + 3b_{0,1,2}\, c_{1,0}\, c_{0,1}^2\,, \qquad (2.20)
\end{aligned}
$$

$$
\begin{aligned}
c_{0,4} &= b_{0,4,0} + b_{0,1,1}\, c_{0,3} + b_{0,2,1}\, c_{0,2} + b_{0,3,1}\, c_{0,1} + \\
&+ b_{0,2,2}\, c_{0,1}^2 + 2b_{0,1,2}\, c_{0,1}\, c_{0,2} + b_{0,1,2}\, c_{0,1}^3 \qquad (2.21)
\end{aligned}
$$

. . . .

Equations (2.8) – (2.21) allows us to obtain, in turn, the coefficients c_{ij} of the series (2.2) for the analytic implicit function $f(x, y)$.
If in series (1.5), we have $a_{0,0,1} \neq 0$ but $a_{0,0,1} \neq 1$, then in equations (2.8) – (2.21) we must replace $b_{i,j,k}$ by $\dfrac{b_{i,j,k}}{a_{0,0,1}}$.
Let

$$
M = \sup\{\, |B(x, y, z)| : |x| \leq r_1,\ |y| \leq r_2,\ |z| \leq r_3 \}\,, \qquad (2.22)
$$

be, where $B(x, y, z)$ is given by (2.6) and define for all $(x, y, z) \in \mathbb{R}^3$ with $|x| \leq r_1$, $|y| \leq r_2$, $|z| \leq r_3$ the function

$$
G(x, y, z) = M \left\{ \left(1 - \frac{x}{r_1}\right)^{-1} \left(1 - \frac{y}{r_2}\right)^{-1} \left(1 - \frac{z}{r_3}\right)^{-1} - 1 - \frac{z}{r_3} \right\} . \qquad (2.23)
$$

It is easy to see that

$$G(x,y,z) = M\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{x^i y^j}{r_1^i r_2^j} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=2}^{\infty}\frac{x^i y^j z^k}{r_1^i r_2^j r_3^k}\right\}, \tag{2.24}$$

and by Lemma 1.1, we see that the series (2.24) is a majorant of the series (2.6), so (2.6) is absolutely convergent for $|x| \le r_1$, $|y| \le r_2$, $|z| \le r_3$.
The equation

$$z = G(x,y,z)\,, \tag{2.25}$$

equivalent to (1.5), can be solved explicitly for z and obtain the implicit function $z = f(x,y)$.
To solve (2.25) for z, from (2.23) we obtain

$$\frac{z}{M} = \left(1-\frac{x}{r_1}\right)^{-1}\left(1-\frac{y}{r_2}\right)^{-1}\left(1-\frac{z}{r_3}\right)^{-1} - 1 - \frac{z}{r_3}\,,$$

and after some computation, one has

$$z^2 - \frac{r_3^2}{r_3+M}\,z + \frac{Mr_3^2}{r_3+M}\left(1-\left(1-\frac{x}{r_1}\right)\left(1-\frac{y}{r_1}\right)\right) = 0 \tag{2.26}$$

We take $r = \min\{r_1, r_2\}$ and consider the ball $\overline{B}_2(\mathbf{0}) \subset \mathbb{R}^2$. Then the function $z = f(x,y)$, where $f : \overline{B}_2(\mathbf{0}) \longrightarrow \mathbb{R}$, given by (2.26) is the analytic implicit function defined by the equation (1.5) and having the power series (2.2) with coefficients c_{ij} obtained from the equation (2.8) – (2.21).

3 Application

Let the equation (1.4) be, with

$$\begin{aligned} F(x,y,z) &= -x + xz + yz + \frac{1}{2!}\,x^2 z + xyz + \frac{1}{2!}\,y^2 z + \\ &+ \frac{1}{3!}\,x^3 z + \frac{1}{2}\,x^2 yz + \frac{1}{2}\,xy^2 z + \frac{1}{3!}\,y^3 z + \ldots \end{aligned} \tag{3.1}$$

The series (3.1) satisfies the hypothesis of the theorem 2.1 and thus there exists an analytic implicit function $f(x,y)$ satisfying the equation

$$f(x,y,f(x,y)) = 0\,, \quad \forall\, x,y \in \mathbb{R}\,,$$

and the function f has the power series (2.2) with coefficients given by the equations (2.8) – (2.21).
In our case, we obtain

$$c_{1,0} = 1, \quad c_{0,1} = 0, \quad c_{2,0} = -1, \quad c_{1,1} = -1, \quad c_{0,2} = 0,$$

$$c_{3,0} = \frac{1}{2}, \quad c_{2,1} = 1, \quad c_{1,2} = \frac{1}{2}, \quad c_{0,3=0},$$

$$c_{4,0} = \frac{-1}{6}, \quad c_{3,1} = -\frac{1}{2}, \quad c_{2,2} = -\frac{1}{2}, \quad c_{0,4} = -\frac{1}{6}, \dots$$

The power series of analytic implicit function f will be

$$\begin{aligned} f(x,y) &= x - x^2 - xy + \frac{1}{2}x^3 + x^2y + \frac{1}{2}xy^2 - \\ &- \frac{1}{6}x^4 - \frac{1}{2}x^3y - \frac{1}{2}x^2y^2 - \frac{1}{6}xy^3 + \dots \end{aligned} \tag{3.2}$$

We see that

$$f(x,y) = x\{1 - (x+y) + \frac{1}{2}(x+y)^2 - \frac{1}{6}(x+y)^3 + \cdots\},$$

i.e.

$$f(x,y) = x\,e^{-(x+y)} \tag{3.3}$$

We also see that the sum of the series (3.1) is

$$F(x,y,z) = z\,e^{x+y} - x, \quad x,y,z \in \mathbb{R} \tag{3.4}$$

and then we have

$$F(x,y,f(x,y)) = x\,e^{-(x+y)}e^{x+y} - x = 0, \quad \forall\, x,y \in \mathbb{R}.$$

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The aim of this article is to state and to proof a version of analytic implicit function theorem for functions of two variables. Then we will give an application.

Received: 07.11.2002

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