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A TAYLOR TYPE THEOREM WITH LATERAL DERIVATIVE

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In [1] is proved a Lagrange type theorem regarding lateral derivatives. Our aim is to formulate a Taylor type result which deals with lateral derivative, and to present an application under convexity assumption.

Theorem 1 (Lagrange type) See [1].

If the function $f:[a,b]\to R$

(i) *is continuous on* $[a,b]$

(ii) *has finite left-hand derivative* $f_-^{'}(x)$ *at each* $x\in]a,b]$,

then there exist $\xi_1,\xi_2\in]a,b]$ *such that*

$$f_-^{'}(\xi_1)\le\frac{f(b)-f(a)}{b-a}\le f_-^{'}(\xi_2).$$

Remark. If we take right-hand derivative instead of left-hand derivative, an analogous of the Theorem 1 holds.

Remark. The following function shows that admitting an $x_0\in]a,b]$ where f is only left-hand continuous, the conclusion of the Theorem 1 is not preserved.

$$f:[-1,1]\to R,\quad f(x)=\begin{cases}x+1,\ x\in[-1,0]\\ x-1,\ x\in(0,1]\end{cases}$$

Remark. Next function shows that supposing continuity only on $]a,b]$ instead of $[a,b]$, the conclusion of the Theorem 1 is not preserved.

$$f:[-1,1]\to R,\quad f(x)=\begin{cases}0,\ x=0\\ x,\ x\in]-1,1]\end{cases}$$

The following theorem is a Taylor type result with a left-hand derivative. We will prove it by using Theorem 1.

Theorem 2 (Taylor type)

Let $I \subset R$ be an open interval. If the function $f : I \to R$ is n times differentiable on I and

(i) $f^{(n)}$ is continuous on I

(ii) $f^{(n)}$ has finite left-hand derivative $\left(f^{(n)}\right)'_-(x)$ at each $x \in I$,

then for every $x_0, x \in I$, $x_0 \neq x$ there exist $\eta_1, \eta_2 \in]x, x_0]$ or $\eta_1, \eta_2 \in [x_0, x[$ such that the coefficient A defined by the relation

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}\cdot(x-x_0) + \frac{f''(x_0)}{2!}\cdot(x-x_0)^2 + \dots$$
$$\dots + \frac{f^{(n)}(x_0)}{n!}\cdot(x-x_0)^n + A\cdot(x-x_0)^{n+1} \tag{1}$$

is delimited by

$$\frac{\left(f^{(n)}\right)'_-(\eta_1)}{(n+1)!} \le A \le \frac{\left(f^{(n)}\right)'_-(\eta_2)}{(n+1)!} \tag{2}$$

Proof. Let I be an open interval of R. Let $f : I \to R$ be n times differentiable such that satisfies (i) and (ii).

For $x_0, x \in I$ fixed let A be the coefficient defined by the equality (1).

Consider $\varphi : I \mapsto R$,

$$\varphi(t) = f(t) + \frac{f'(t)}{1!}\cdot(x-t) + \frac{f''(t)}{2!}\cdot(x-t)^2 + \dots + \frac{f^{(n)}(t)}{n!}\cdot(x-t)^n + $$
$$+ A\cdot(x-t)^{n+1}.$$

The properties of the function f imply that φ is continuous on I and has finite left-hand derivative $\varphi'_-(t)$ at each $t \in I$.

By a simple computation we get

$$\varphi'_-(t) = f'(t) + \frac{f''(t)}{1!}\cdot(x-t) - \frac{f'(t)}{1!} + \frac{f'''(t)}{2!}\cdot(x-t)^2 - \frac{f''(t)}{1!}\cdot(x-t) + \dots$$
$$\dots + \frac{f^{(n)}(t)}{(n-1)!}\cdot(x-t)^{n-1} - \frac{f^{(n-1)}(t)}{(n-2)!}\cdot(x-t)^{n-2} + \frac{\left(f^{(n)}\right)'_-(t)}{n!}\cdot(x-t)^n -$$
$$-\frac{f^{(n)}(t)}{(n-1)!}\cdot(x-t)^{n-1} - A\cdot(n+1)\cdot(x-t)^n =$$
$$= \left(\frac{\left(f^{(n)}\right)'_-(t)}{n!} - (n+1)\cdot A\right)\cdot(x-t)^n, \forall t \in I.$$

We have $\varphi(x) = f(x)$; (1) implies $\varphi(x_0) = f(x)$. Thus $\varphi(x_0) = \varphi(x)$.

Apply Theorem 1 for the function φ on the interval $[x_0, x]$ in case of $x_0 < x$. It results that there exist $\xi_1, \xi_2 \in]x_0, x]$ such that

$$\varphi'_-(\xi_1) \le \frac{\varphi(x) - \varphi(x_0)}{x - x_0} \le \varphi'_-(\xi_2).$$

Since $\varphi(x_0) - \varphi(x) = 0$, this means that

$$\left(\frac{\left(f^{(n)}\right)'_-(\xi_1)}{n!} - (n+1)\cdot A\right)\cdot(x-\xi_1)^n \le$$
$$\le 0 \le \left(\frac{\left(f^{(n)}\right)'_-(\xi_2)}{n!} - (n+1)\cdot A\right)\cdot(x-\xi_2)^n. \tag{3}$$

Since $(x-\xi_1)^n > 0$ and $(x-\xi_2)^n > 0$, dividing by $(n+1)$ we obtain inequalities (2) with $\eta_1 = \xi_1, \eta_2 = \xi_2$.

Apply Theorem 1 for the function φ on the interval $[x, x_0]$ in case of $x < x_0$. It results that there exist $\xi_1, \xi_2 \in]x, x_0]$ such that (3) holds. In this case $x - \xi_1 < 0$, $x - \xi_2 < 0$. For n even we obtain (2) with $\eta_1 = \xi_1, \eta_2 = \xi_2$. For n odd we obtain (2) with $\eta_1 = \xi_2, \eta_2 = \xi_1$.

Remark. Dealing with right-hand derivative instead of left-hand derivative, an analogous of the Theorem 2 can be formulated.

Since a convex function is continuous and possess finite left-hand derivative (as well as finite right-hand derivative) at every interior point of its domain, the following result holds as an immediate consequence of Theorem 2.

Corollary 1. *Let $f : I \to R$ be n times differentiable on the open interval $I \subset R$. If the function $f^{(n)} : I \to R$ is convex, then for every $x_0, x \in I$, $x_0 \ne x$ there exist $\eta_1, \eta_2 \in]x, x_0]$ or $\eta_1, \eta_2 \in]x_0, x]$ such that the coefficient A defined by the Taylor type relation (1) verifies inequalities (2).*

Since the left-hand derivative of a convex function (as well as its right-hand derivative) is increasing on the interior of the domain of the function, the following corollary holds.

Corollary 2. *Let the function $f : I \to R$ be n times differentiable on the open interval $I \subset R$. If the function $f^{(n)} : I \to R$ is convex, then for every $x_0, x \in I$, $x_0 \ne x$ the coefficient A defined by the Taylor type relation (1) satisfies*

$$\frac{\left(f^{(n)}\right)_-'(x_0)}{(n+1)!} \le A \le \frac{\left(f^{(n)}\right)_-'(x)}{(n+1)!}, \quad \text{if } x_0 < x$$

respectively

$$\frac{\left(f^{(n)}\right)_-'(x)}{(n+1)!} \le A \le \frac{\left(f^{(n)}\right)_-'(x_0)}{(n+1)!}, \quad \text{if } x < x_0 .$$

In addition, if $a, b \in I$ *such that* $x_0, x \in [a,b]$, *then*

$$\frac{\left(f^{(n)}\right)_-'(a)}{(n+1)!} \le A \le \frac{\left(f^{(n)}\right)_-'(b)}{(n+1)!}.$$

Remark. Taking right-hand derivative instead of left-hand derivative in these corollaries, we obtain analogous results.

References

1. Cobzaş, Şt., *Mathematical Analysis (Differential Calculus)*, Presa Universitară Clujană, Cluj, 1997, in Romanian
2. Flett, M. F., *Differential Analysis*, Cambridge, 1980

Abstract. In this note we formulate a Taylor type result which deals with a lateral derivative; see Theorem 2. Then we give an application under convexity assumption; see Corollary 2.

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