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ON THE ORDER OF CONVERGENCE OF SOME SEQUENCES OF RIEMANN'S SUMS OF FUNCTIONS WITH CONVEX DERIVATIVE

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Denote I an open interval, $I \subset R$. Let $f : I \to R$ be a differentiable function with $f' : I \to R$ convex.

Since f is continuous on I, it is integrable on every $[a,b] \subset I$.

So, the sequence of Riemann's sums

$$T_n = \frac{b-a}{n}\sum_{i=1}^{n} f\left(a+(2i-1)\cdot\frac{b-a}{2n}\right),\ n \in N^*$$

is convergent and

$$\lim_{n\to\infty} T_n = \int_a^b f(x)dx .$$

Next we will characterize the convergence order of the sequence (T_n).

Theorem. *Let $I \subset R$ be an open interval, $f : I \to R$ a differentiable function with $f' : I \to R$ convex, and $a, b \in I$, $a < b$. Then*

$$\lim_{n\to\infty} n^2\left[\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+(2i-1)\cdot\frac{b-a}{2n}\right) - \int_a^b f(x)dx\right] = \frac{(b-a)^2\left[f'(a)-f'(b)\right]}{24}.$$

Remark. This formula is well-known for an other class of functions, namely for f twice differentiable with f'' integrable on $[a,b]$; see [1], Part II, Chap. 1, §2.

Proof. Denote

$$\overline{T}_n = T_n - \int_a^b f(x)dx,\ c_i = a + i\cdot\frac{b-a}{n},\ d_i = a+(2i-1)\cdot\frac{b-a}{2n}.$$

Then we have

$$\overline{T}_n = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} [f(d_i) - f(x)] dx .$$

Since the function f' is convex, it is continuous on I and it possess finite left-hand derivative $(f')'_-(x)$ at each $x \in I$. So, for the function f we can apply a Taylor type theorem regarding left-hand derivative; see [2].

It results that for $x \in [c_{i-1}, c_i]$ fixed, there exist $\eta_1, \eta_2 \in]d_i, x]$ or $\eta_1, \eta_2 \in]x, d_i]$ such that the coefficient A determined by the equality

$$f(x) = f(d_i) + \frac{f'(d_i)}{1!} \cdot (x - d_i) + A \cdot (x - d_i)^2$$

is delimited by

$$\frac{(f')'_-(\eta_1)}{2!} \le A \le \frac{(f')'_-(\eta_2)}{2!}$$

and taking into account that the left-hand derivative of a convex function is increasing on the interior of the domain of the function, it results that

$$\frac{(f')'_-(c_{i-1})}{2!} \le A \le \frac{(f')'_-(c_i)}{2!} .$$

So, for all $x \in]c_{i-1}, c_i[$ we have

$$f'(d_i) \cdot (x - d_i) + \frac{(f')'_-(c_{i-1})}{2!} \cdot (x - d_i)^2 \le f(x) - f(d_i) \le$$

$$\le f'(d_i) \cdot (x - d_i) + \frac{(f')'_-(c_i)}{2!} \cdot (x - d_i)^2 .$$

A simple computation shows

$$\int_{c_{i-1}}^{c_i} (x - d_i) dx = 0$$

$$\int_{c_{i-1}}^{c_i} (x - d_i)^2 dx = \frac{(b-a)^3}{12 \cdot n^3} .$$

Then for $\overline{T}_n$ results the following delimitation

$$\frac{(a-b)^3}{24n^3} \sum_{i=1}^{n} (f')'_-(c_i) \le \overline{T}_n \le \frac{(a-b)^3}{24n^3} \sum_{i=1}^{n} (f')'_-(c_{i-1}) \qquad (1)$$

Since the function $(f')'_-$ is monotonic, it is integrable on $[a,b]$.

Since $(f')'_-$ is integrable on $[a,b]$, we have

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}(f')'_-(c_{i-1})=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}(f')'_-(c_i)=\int_a^b(f')'_-(x)dx.$$

A convex function is continuous on the interior of its domain and has finite derivative at each interior point of its domain except at most a countable set of interior points. Since I is open interval, the convex function f' is continuous on I and $(f')'(x)$ exists and is finite for all $x\in I-S$, where S, $S\subset I$, is at most countable.

Obviously, on $I-S$ the function f' is a primitive of $(f')'$ and $(f')'(x)=(f')'_-(x)$ for all $x\in I-S$. Since S is at most countable, the continuous function f' is a generalized primitive of $(f')'_-$ on I.

For integrable functions, which possess generalized primitives, the Newton-Leibniz formula holds; see [3]. So we have

$$\int_a^b(f')'_-(x)dx=f'(b)-f'(a)$$

Now, based on the inequalities (1) and on the limits

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}(f')'_-(c_{i-1})=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}(f')'_-(c_i)=f'(b)-f'(a)$$

we obtain

$$\lim_{n\to\infty}n^2\cdot\overline{T}_n=\frac{(b-a)^2[f'(a)-f'(b)]}{24}.$$

References

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Abstract. Let $I \subset R$ be an open interval, $f: I \to R$ a differentiable function with $f': I \to R$ convex, and $a, b \in I$, $a < b$.

In this note, by using a Taylor-type result regarding left-hand derivative, we prove the following formula

$$\lim_{n\to\infty} n^2 \left[\frac{b-a}{n} \sum_{i=1}^{n} f\left(a + (2i-1) \cdot \frac{b-a}{2n} \right) - \int_a^b f(x)dx \right] = \frac{(b-a)^2 \left[f'(a) - f'(b) \right]}{24}$$

which is well-known for an other class of functions, see [1], Part II, Chap. 1, §2, namely for f twice differentiable with f'' integrable on $[a, b]$.

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