Lucrarile Seminarului de
CREATIVITATE MATEMATICĂ
Vol. 11(2002), 113 -124

ON THE WATER-ROCKET ENGINE PROBLEM. A CASE STUDY IN MATHEMATICAL MODELING

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1 Introduction

A rocket is essentially a tank that is propelled upwards by the ejection of some mass. The acceleration of the ejected material produces an equal reaction in the opposite direction by pushing the matter downwards, out of a rocket exhaust produces a reaction that tries to push the rocket upwards. During the course of the flight, the rockets momentum keeps it going until the drag and the gravity overcome it. In the case of the water rocket, the reaction mass is water and the force that is used to push it out comes from the pressure of the compressed air inside the rocket. Water rockets use a dense reaction mass (water is very heavy, as you can see by filling a bucket of water), which is ejected at high speed by the energy stored by pressurized inert air. This generates enough driving force to fire the rocket (which being usually plastic bottles are very light) into the air at high speed. The complicated balancing act between the reaction mass and the rocket mass is more complex in water rockets. This is because the rocket has a fixed volume and increasing the amount of water means decreasing the amount of volume available for the pressurized air and hence less stored energy. Also, if one adds to much water into the rocket, the rocket will weight more and, consequently, there will be less 'stored energy'. This makes the rocket slower to launch. Therefore it achieves a lower altitude. On the other hand, too little water inside the rocket could imply that the rocket may not even get off the ground. A simulation example due to M. Sullivan may be found at http://webalt.markworld.com/altitude_example.html. This web-page drives a rocket simulation program which will project the peak altitude

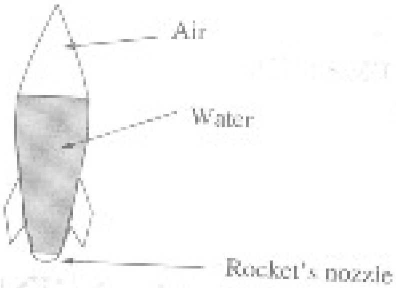

Figure 1: *A water-rocket engine*

and other flight profile information about a model rocket. One may find Internet resources on water-rocket flight simulation. See, e.g., the home page of National Physical Laboratory Teddington, UK.

It is clear that the distances and speeds traveled with such a rocket model will not be able to simulate, for instance, the twins paradox (cf. [4], p. 11), but nevertheless, it brings nicely together several important fluid dynamics principles involved in the rocket's flight. One of these principles which we aim to point out and then to apply, is the Bernoulli 's law[1]. However, we do not attempt neither to derive the Bernoulli 's equation nor to specify its range of validity. We restrict to mention only that for the corresponding deduction one uses the continuity equation[2] and the conservation of energy for a 'particular' fluid flow inside the rocket. These facts should be read by the student himself from the literature. More about Bernoulli' s law and its impact in simplifying many intricate problems of fluid dynamics, see, e.g., [11] (textbook with many examples), [1] (more theoretical approach) and [6] (a bit of philosophy). In [1], [10] one shows how powerful can become the 'tools' offered by the complex analysis in the plane fluid mechanics. When applying these two physical laws[3] to the fluid at two *arbitrary* distinct positions, one obtains

$$p_1 - p_2 = \frac{1}{2}\left(v_2^2 - v_1^2\right) + \rho g(y_2 - y_1). \tag{1}$$

This is one way to write Bernoulli 's equation. In Eq. (1), p_1, p_2 denote fluid pressures, g is the acceleration of gravity, ρ is the mass density of the fluid and the numbers y_1, y_2 represent the two spatial positions where the pressures or/and the velocities were measured. At this moment we should specify, that we have considered a one-dimensional flow. Nevertheless, the

[1] Cf. e.g. [13], [12], in most textbooks of science and technology, the explanation of the aerodynamic lifting force is based on Berboulli's law. This explanation has fundamental drawbacks, and its reasoning is incomplete and often wrong. However, we will not highlight these facts within this text.

[2] The volume of a fluid that flows through a (closed) surface, e.g. a section of a tube, must be equal to the volume of fluid that flows out in any given time interval.

[3] See also Problem 7.

application of Bernoulli 's law is not restricted to one space dimension. In this form, it states that the pressure change has a component due to the change in the velocity of the fluid and a component due to the change in elevation. However, we suggest also the form:

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2, \quad (2)$$

or, concluding,

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = constant. \quad (3)$$

We illustrate the use of the Bernoulli' s result by means of an example:

Problem: Consider, a bucket h ($h > 0$) meters high, filled with water. The bucket has a small hole in the bottom. What is the velocity of the water as it leaves the bucket?

Hint: We apply Bernoulli 's equation as expressed above Eq. (3). The two places we want to consider are the top of the bucket and at the hole in the bottom of the bucket. Let us consider $y = 0$ at the bottom of the bucket and thus $y = h$ is at the top of the bucket. Thus we get the relation:

$$p_t + \frac{1}{2}\rho v_t^2 + \rho g h = p_b + \frac{1}{2}\rho v_b^2.$$

v_b is what we want to solve for. We have denoted here by v_t = top-fluid velocity and by v_b = bottom-fluid one. Since the whole in the bucket is a small one, the bucket will not drain fast and we can consider the fluid at the top of the bucket to have zero velocity, i. e. $v_t = 0$. Also, since the top of the bucket is open to atmospheric pressure, and so is the hole in the bottom of the bucket. It means: $p_t = p_b = p$. Thus we are left with:

$$\frac{1}{2}\rho v_b^2 = \rho g h$$

or

$$v_b = \sqrt{2gh}.$$

The latter result is called *Torricelli 's law*. It states that the velocity of the fluid emerging from a reservoir increases as the square of the height of the reservoir.

Note: Bernoulli 's equation shows clearly that: When the velocity of a fluid increases, the pressure drops. This is called Venturi effect.

2 Elementary modeling of the direct problem

Let us consider $V_0 = V(t_0)$ the volume occupied by the water content at the initial time $t = t_0$ and let us denote by $V(t)$ the volume drained out from the inner part of the rocket at the time $t \geq t_0$ under the air pressure $p(t)$. The remaining water will occupy the volume

$$V_w(t) = V_0 - V(t) \geq 0. \quad (4)$$

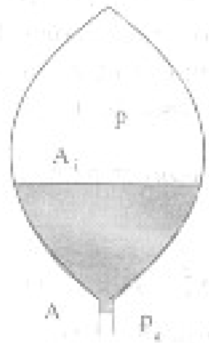

Figure 2: *The flying bucket*

2.1 Derivation of the equations governing the model

We begin with the derivation of the model equations. We may consider, from one side, the adiabatic expansion of the gas, i.e. the pressure acting on the air volume inside the rocket engine, respects the following physical law:

$$pV^{\chi} = constant = c > 0, \tag{5}$$

where $\chi = \frac{C_p}{C_v}$ represents the adiabatic expansion coefficient and C_p, C_v are the specific heat capacities at constant pressure and at constant volume, respectively. χ is usually fixed at 1.4. The expansion of the compressed gas in the rocket is its main energy source. We assume that the gas expands adiabatically, i.e. it expands in such a short time frame that heat cannot flow fast enough from the outside world into the gas to keep it at a constant temperature. In the sequel, the gas cools as it expands. From the other side, if the ratio of the cross-sectional areas A and A_i, $\left(\frac{A}{A_i}\right)^2 \ll 1$, is neglected, then one may apply the Bernoulli 's law in the form written below:

$$p_a + \frac{1}{2}\rho v_a^2 = p. \tag{6}$$

We denote $\rho = \rho_{H_2O}$ the density of the pure (i.e. de-ionized) water, p_a denotes the atmospheric pressure and v_a is the velocity of the draining water taken into account at the local coordinate of the evacuation drain.

It yields

$$v_a(t) = \sqrt{\frac{2(p(t) - p_a(t))}{\rho}}, \; t \geq t_0. \tag{7}$$

The volume of the expelled water during the time-interval $[t_0, t_{out}]$, $t_{out} \geq t_0$ may be estimated as follows:

$$\frac{dV}{dt}(t) = v_a(t)A = \sqrt{\frac{2(p(t) - p_a(t))}{\rho}}A, \quad V(t_0) = V_0. \tag{8}$$

The subscript 'out' indicates a property at the rocket 's outlet. See also [2]. The first equality in Eq. (8) is obtained by applying the mass conservation principle to a fluid control volume inside the rocket 's water content. The concrete deduction is left to the reader.

Using conveniently Eq. (5), it results

$$\frac{1}{p}\frac{dp(t)}{dt} + \chi\frac{1}{V}\frac{dV(t)}{dt} = 0,\ \forall t \in]t_0, t_{out}[. \tag{9}$$

Therefore, it holds

$$\frac{dp(t)}{dt} + \chi c^{-\frac{1}{\chi}} p(t)^{\frac{\chi+1}{\chi}} \sqrt{\frac{2(p(t)-p_a(t))}{\rho}}A = 0 \text{ and } V(t_0) = V_0. \tag{10}$$

On this way, one obtains the differential equation for the volume variation:

$$\frac{dV(t)}{dt} = \sqrt{\frac{2\left(\frac{c}{V(t)^{\chi}} - p_a(t)\right)}{\rho}}A,\ \forall t \in]t_0, t_{out}[\text{ and } V(t_0) = V_0. \tag{11}$$

2.1.1 Vertical launching

Rocket propulsion works according to the Newton 's third law of motion: For every action, there is an equal and opposite reaction. A rocket engine ejects something out of its thrust nozzle, and the action of pushing that something backwards causes an equal and opposite reaction that pushes the rocket forwards. This 'something' being ejected is called the reaction mass. This principle is the same for all the rockets: From water rockets launched in the park (reaction mass = water) to the high-tech ion engines that propel modern deep-space probes (reaction mass = tiny ionized particles).

To find out the force generated by the ejection of the reaction mass we have to use the Newton 's second law of motion, which states that the force is proportional to the rate of change of the momentum (mass times velocity). Thus one gets:

$$F(t) = \rho\frac{dV(t)}{dt}v_a(t) = \rho\left(\sqrt{\frac{2\left(\frac{c}{V^{\chi}} - p_a\right)}{\rho}}\right)^2 A = 2A\left(\frac{c}{V^{\chi} - p_a}\right) \tag{12}$$

$$\forall t \in]t_0, t_{out}[.$$

One should also note that in a vertical system, the interface between the water and the air may be considered as a piston and any gas dissolving in water is ignored, e.g. CO_2-gas will dissolve and reduce the pressure in the rocket until it reaches equilibrium and if there is enough, it will effervesce as it de-pressurizes upon passing through the nozzle making the viscosity and the other flow characteristics quite unpredictable.

The total mass of the rocket is given by:

$$m(t) = \rho V_w(t) + m_0 + m_a = \rho (V_0 - V(t)) + m_0, \ \forall t \in [t_0, t_{out}], \tag{13}$$

where m_0 represents the mass of the empty rocket, m_a is the mass of compressed air trapped in the rocket chamber. In this approach we consider $m_a \approx 0$.

Therefore, it yields

$$a(t) = \frac{F(t)}{m(t)} = \frac{2A\left(\frac{c}{V^{\gamma}(t)} - p_a\right)}{m_0 + \rho (V_0 - V(t))}, \forall t \in [t_0, t_{out}]. \tag{14}$$

We denote x_1, x_2 and x_3 the vertical coordinate, the corresponding velocity (at the point x_1) and the volume of the already drained out water, respectively. In occurrence , we get the following differential system:

$$\frac{dx_1(t)}{dt} = v(t) = x_2(t), \tag{15}$$

$$\frac{dx_2(t)}{dt} = a(t) - g = \frac{2A\left(\frac{c}{x_3^{\gamma}(t)} - p_a(t)\right)}{m_0 + \rho (V_0 - x_3(t))} - g, \tag{16}$$

$$\frac{dx_3(t)}{dt} = \frac{dV(t)}{dt} = A\sqrt{\frac{2\left(\frac{c}{x_3^{\gamma}(t)} - p_a(t)\right)}{\rho}}. \tag{17}$$

One associates to the previous system Eqs. (15) - (17), the initial values $x_1(t_0) = x_{10}$, $x_2(t_0) = x_{20}$ and $x_3(t_0) = x_{30}$.

2.1.2 Slanting launching

We introduce the orthogonal projections a_x, a_y of the draining water acceleration a onto the coordinate frame as follows:

$$a_x = \frac{v_x}{\sqrt{v_x^2 + v_y^2}} |a|, \tag{18}$$

and

$$a_y = \frac{v_y}{\sqrt{v_x^2 + v_y^2}} |a|. \tag{19}$$

We easily obtain the differential system:

$$\frac{d^2x}{dt}(t) = \frac{\frac{dx}{dt}(t)}{\sqrt{\left(\frac{dx}{dt}(t)\right)^2 + \left(\frac{dy}{dt}(t)\right)^2}} \frac{2A\left(\frac{c}{V(t)^{\gamma}} - p_a(t)\right)}{m_0 + \rho (V_0 - V(t))}, \tag{20}$$

$$\frac{d^2y}{dt}(t) = \frac{\frac{dy}{dt}(t)}{\sqrt{\left(\frac{dx}{dt}(t)\right)^2 + \left(\frac{dy}{dt}(t)\right)^2}} \frac{2A\left(\frac{c}{V(t)^\chi - p_a(t)}\right)}{m_0 + \rho\left(V_0 - V(t)\right)}, \tag{21}$$

$$\frac{dV}{dt}(t) = \sqrt{\frac{2\left(\frac{c}{V(t)^\chi} - p_a(t)\right)}{\rho}} A, \ \forall t \in]t_0, t_{out}[. \tag{22}$$

In order to use MATLAB procedures to solve this system numerically, one must transform it, first of all, in a first-order differential system.

Thus, introducing the new variables,

$$z_1(t) = x(t), \ z_2(t) = y(t), \ z_3(t) = \frac{dx}{dt}(t), \ z_4(t) = \frac{dy}{dt}(t), \ z_5(t) = V(t), \tag{23}$$

one obtains immediately the nonlinear coupled system of first-order ODEs:

$$\frac{dz_1}{dt}(t) = z_3(t), \tag{24}$$

$$\frac{dz_2}{dt}(t) = z_4(t), \tag{25}$$

$$\frac{dz_3}{dt}(t) = \frac{z_3(t)}{\sqrt{z_3^2(t) + z_4^2(t)}} \frac{2A\left(\frac{c}{z_5^\chi(t)} - p_a\right)}{m_0 + \rho\left(V_0 - z_5(t)\right)}, \tag{26}$$

$$\frac{dz_4}{dt}(t) = \frac{z_4(t)}{\sqrt{z_3^2(t) + z_4^2(t)}} \frac{2A\left(\frac{c}{z_5^\chi(t)} - p_a\right)}{m_0 + \rho\left(V_0 - z_5(t)\right)} - g. \tag{27}$$

$$\frac{dz_5}{dt}(t) = \frac{2\left(\frac{c}{z_5^\chi(t)} - p_a\right)}{\rho} A. \tag{28}$$

In order to complete the model, one should specify the initial conditions $z_1(t_0) = z_{10}$, $z_2(t_0) = z_{20}$, $z_3(t_0) = z_{30}$, $z_4(t_0) = z_{40}$ and $z_5(t_0) = z_{50}$.

2.2 Activities

Problem 1 What is the physical meaning of Eq. (5)? Precise the S. I. units for χ.

Problem 2 Derive (from modeling investigations) Eqs. (6)-(11) and the differential systems (15)- (17) and (24)-(28). Does Eq. (6) violate the linear momentum conservation?

Problem 3 For $p(t_0) = p_0$, $V(t_0) = V_0$, $x(t_0) = (x_{10}, x_{20}, x_{30})$ given, provide existence results for the ODEs (10), (11), and also, for the systems of ODEs (15)- (17) and (24)-(28).

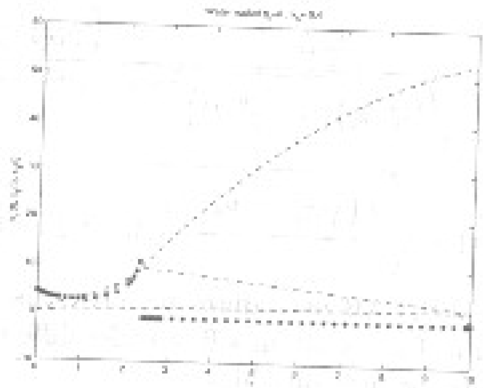

Figure 3: *An example of vertical launching.*

Problem 4 Use the MATLAB procedure **ode45** in order to solve numerically the system (15)- (17) for $p_a = 1, p_0 = 8, \chi = 1.4, t_0 = 0, V_0 = 0.4, A = 0.05, \rho = 0.2, g = 1, m_0 = 0.05$. *This is the case of the vertical launching.* Interpret the numerical results you have obtained.

Hint: See [9] and the MATLAB 's help resources.

Problem 5 Use the MATLAB procedure **ode45** in order to solve numerically the system (24)-(28) for $p_a = 1, p_0 = 8$, $\chi = 1.4, t_0 = 0$, $V_0 = 0.4$, $A = 0.05, \rho = 0.2, g = 1, m_0 = 0.05, v = 0.2$ (the initial speed), $\alpha = 1$(launching angle), $t_0 = 10$(the observation time). *This is the case of the slanting launching.* Interpret the numerical results you have obtained.

Problem 6 Compute the volume of a rocket which has an ellipsoidal form given by the relation:

$$(R): \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1, \quad a, b, c \in \mathbb{R}_+^* \text{ and } (x, y, z) \in \mathbb{R}^3.$$

How many kilograms water may enter in such a device? But gasoline?

Problem 7 What are the validity limits of the Bernoulli 's law in the present case of water-rocket engine?

Problem 8 1. Derive the nonlinear integral equation of Volterra type

$$V(t) = V(t_0) + A\int_{t_0}^{t} \sqrt{2\left(p(0)\left[\frac{V_0}{V(t)}\right]^{\chi} - p_a\right)\frac{1}{\rho}}\,dt; \qquad (29)$$

2. Give an interval of existence of the solution of Eq. (8). Afterwards, solve the equation iteratively.

Problem 9 1. Can Torricelli 's law be applied in order to compute v_a? Explain in which conditions. What would happen in this situation with the rocket engine?

2. Consider now a conical rocket at rest having the cross-sectional radius at a height y exactly $y \tan \alpha$. Find out the emptying time t_{out} for this rocket.

   Hint: Apply the Torricelli 's law. Compare to [2], p. 42-43.

3. Using the 'idea' from the previous request, find out a general computation scheme for the emptying time t_{out} in the case of an arbitrary symmetrical rocket with prescribed cross-section $S = S(y)$.

Hint for 3). Have a look on Section 4.3 from [2].

Problem 10 Let us consider the volume:

$$V_n(t) = \begin{cases} \exp(\int_0^t A \left(\frac{\sin x}{x}\right)^n dx), & t \neq 0 \\ \exp(1)t + n^2, & t = 0 \end{cases}, n \in \mathbb{N},$$

- Compute v_a;
- Evaluate the limits

$$\lim_{n\to\infty} \lim_{t\to\infty} V_n(t), \quad \lim_{t\to\infty} \lim_{n\to\infty} V_n(t) \text{ and } \lim_{t\to 0} \frac{d}{dt} V_n(t).$$

Problem 11 In this problem the air is considered as being an ideal gas. Discuss the case when the air is a real gas.

Problem 12 Check the dimensionality of all the relations appearing within text. Provide a nomenclature for all relevant physical quantities. This should contain the next columns: *notations*, *S. I. dimensions* and *constants* (if there is the case).

Problem 13 What would happen if instead of water we would have a highly viscous liquid (e.g. oil, paraffin, honey etc.)?

Problem 14 1. What is the tendency of the value of the maximum height reached by the rocket when replacing air by:

- Helium gas;
- Xenon gas.

Note: $\rho_{He} = 0.1785$ g/l, $\rho_{Xe} = 5.887 \pm 0.009$ g/l.

2. What means 'the thrust phase of a rocket'? But 'the coasting phase'?

3. For the case of vertical launching, which initial velocity should have the water-rocket engine in order to arrive at $h_{max} = 20$ meters the maximum height.

Problem 15 What would be the consequences on the rocket movement if the gas temperature increases?

3 Specifications

The work requires the achievement of two basic steps: Firstly, understanding the physical setting (Bernoulli 's law, Torricelli 's law, Venturi 's tube, adiabatic compression of an ideal gas etc.) and then facing the activities. The subject might be assigned to a working group formed by one/two students (for the effective solving of the activities) and a tutor. At the end of the working period, the student should write a LaTex report with his own solutions, and also, provide the corresponding MATLAB codes.

4 Summary

A simplified model of a water rocket was presented. The conventional explanation of aerodynamical lift is based on Bernoulli 's law and velocity differences. Several 'modeling tasks' are drawn and the student is invited to provide convenient solutions as well as new suggestions and improvements.

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Abstract. The purpose of this note is to draw the attention of the undergraduate students in mathematics, physics or engineering sciences to first possible steps in the practice of the mathematical modeling. The proposed subject is an elementary one: A rocket engine, having as fuel only (pure) ideal water, is to be launched. For this to happen, a significant amount of the energy stored in the pressurized gas is converted into kinetic energy of the rocket. See also, e.g., [2], [3], [5]. Several hypothetical activities providing a certain insight to the physical setting are given.
[`Prerequisites`: Elementary fluid mechanics, elementary ordinary differential equations]

Key words: Thrust phase of a rocket, Bernoulli 's law, adiabatic expansion of a gas
2000 AMS subject classification: 65-01, 65L05, 65L12, 65L80, 76-01, 80-01

Received: 25.10.2002

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