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Some applications of integral sums

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Abstract. The correct introduction of definite integrals plays an important role in teaching analysis. This paper is aimed to present one of the possible applications of definite integrals, which could serve the deeper understanding of the basic notion, to show how the basic definition "is working".

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Let us consider (one of) the usual notations for the Riemann integral sums, associated to a given function $f : [a, b] \to R$:

$$\sigma_{\Delta_n}(f, \xi^n) = \sum_{k=1}^{n} f(\xi_k^n)\left(x_k^n - x_{k-1}^n\right)$$

where $\Delta_n = \{x_0^n, x_1^n, x_2^n, ..., x_n^n\}$ is a division of the interval $[a, b]$, $a = x_0^n$, $x_n^n = b$; $\xi_k^n \in \left[x_{k-1}^n, x_k^n\right]$, $k \in \{1, 2, \ldots, n\}$ and $\xi^n = (\xi_1^n, \ldots, \xi_n^n)$.

If the function f is Riemann integrable, we have of course

$$\lim_{\|\Delta_n\| \to 0} \sigma_{\Delta_n}(f, \xi^n) = \int_a^b f(x)dx$$

1. The basic idea of the application

It is usual to introduce different integral sums for the notion of definite integral, which exists if and only if all these sums will be convergent. Conversely, if the integral exists, and f has got a primitive function, the value of the integral computed with the Newton-Leibniz formula will be the limit of all special associated integral sums as well.

As a consequence, for the function $f : [a, b] \to \mathbb{R}$ let us consider the equidistant division of the interval $[a, b]$ in n subintervals of length $\frac{b-a}{n}$, i.e. the points:

$$\Delta_n = \left\{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, a + 3\frac{b-a}{n}, ..., a + (n-1)\frac{b-a}{n}, a + n\frac{b-a}{n}\right\}$$

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Let us form the associated integral sums

$$\sigma_{\Delta_n}(f, \xi^n) = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}$$

where $f(a + k\frac{b-a}{n})$ is the value of the function f in the right-end of the subinterval

$$\left[a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}\right]$$

The limit of these integral sums coincides with the value of the definite integral

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n} = \int_a^b f(x)dx$$

In order to apply the previous result to compute the limits of some special sums, it suffices to identify the function to be integrated and the limits of integration.

2. Application

Example 1. Let us compute the limit:

$$\lim_{n\to\infty}\left(\frac{1+\sqrt{2}+\sqrt{3}+...+\sqrt{n}}{n\sqrt{n}}\right)$$

The given sum can be written step by step:

$$\lim_{n\to\infty}\left(\frac{1+\sqrt{2}+\sqrt{3}+...+\sqrt{n}}{n\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{1+\sqrt{2}+\sqrt{3}+...+\sqrt{n}}{\sqrt{n}}\cdot\frac{1}{n}\right) =$$

$$= \lim_{n\to\infty}\left(\left(\frac{1}{\sqrt{n}}+\frac{\sqrt{2}}{\sqrt{n}}+\frac{\sqrt{3}}{\sqrt{n}}+...+\frac{\sqrt{n}}{\sqrt{n}}\right)\frac{1}{n}\right) =$$

$$= \lim_{n\to\infty}\left(\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\sqrt{\frac{3}{n}}+...+\sqrt{\frac{n}{n}}\right)\frac{1}{n}\right) =$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(0 + k\frac{1}{n}\right)\frac{1}{n}$$

where $f : [0,1] \to \mathbb{R}, f(x) = \sqrt{x}$.
The same time it is known that:

$$\int_0^1 f(x)dx = \int_0^1 \sqrt{x}dx = \frac{2}{3}x\sqrt{x}\Big|_0^1 = \frac{2}{3}$$

As a consequence we can write:

$$\lim_{n\to\infty}\left(\frac{1+\sqrt{2}+\sqrt{3}+...+\sqrt{n}}{n\sqrt{n}}\right)=\int_0^1\sqrt{x}dx=\frac{2}{3}$$

The method which has been sketched may be used in similar way for other cases as well, e.g. if the values in the subintervals are taken in the left-end points, or in a point in the middle of the interval, or if the number of divisions will be $2n$ or 2^n instead of n.

Example 2. Let us compute the limit:

$$\lim_{n\to\infty}\left(\frac{1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+...+\cos\frac{(n-1)\pi}{2n}}{n}\right)$$

Let us rewrite the given sum step by step as follows:

$$\lim_{n\to\infty}\left(\frac{1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+...+\cos\frac{(n-1)\pi}{2n}}{n}\right)=$$

$$=\frac{2}{\pi}\lim_{n\to\infty}\left(\left(1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+...+\cos\frac{(n-1)\pi}{2n}\right)\frac{\pi}{2n}\right)=$$

$$=\frac{2}{\pi}\lim_{n\to\infty}\sum_{k=1}^{n}f\left(0+(k-1)\frac{\pi}{2n}\right)\frac{\pi}{2n}$$

where

$$f:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}, f(x)=\cos x$$

For the last sum the associated integral is:

$$\int_0^{\frac{\pi}{2}}f(x)dx=\int_0^{\frac{\pi}{2}}\cos xdx=\sin x\Big|_0^{\frac{\pi}{2}}=1$$

Finally we have:

$$\lim_{n\to\infty}\left(\frac{1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+...+\cos\frac{(n-1)\pi}{2n}}{n}\right)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos xdx=$$

$$=\frac{2}{\pi}\cdot 1=\frac{2}{\pi}$$

Example 3. Let $f:[0,1]\to R$ be a Riemann integrable function.
Consider the strictly positive sequence $(a_n)_{n\geq 1}$, which has got the following property:

$$\lim_{n\to\infty}\frac{\max(a_1,a_2,...,a_n)}{a_1+a_2+...+a_n}=0$$

We will prove that

$$\lim_{n\to\infty}\sum_{k=1}^{n}f\left(\frac{a_1+a_2+...+a_k}{a_1+a_2+...+a_n}\right)\frac{a_k}{a_1+a_2+...+a_n}=\int_0^1 f(x)dx.$$

Take the sequence $(\triangle_n)_{n\geq 1}$ dividing the interval $[0,1]$, where:

$$\triangle_n=\left(0,\frac{a_1}{a_1+a_2+...+a_n},\frac{a_1+a_2}{a_1+a_2+...+a_n},...,\frac{a_1+a_2+...+a_k}{a_1+a_2+...+a_n},...,1\right),$$

for which, according to the given condition will assure that $\|\triangle_n\| = = \dfrac{\max(a_1,a_2,...,a_n)}{a_1+a_2+...+a_n}\to 0$ when $n\to\infty$.

Let us take the values of the function in the given points, i.e.:

$$\xi_k^n=\frac{a_1+a_2+...+a_k}{a_1+a_2+...+a_n},\ k\in\{1,2,...,n\}$$

It can be seen at once that the associated integral sum is:

$$\sigma_{\Delta_n}(f,\xi^n)=\sum_{k=1}^{n}f\left(\frac{a_1+a_2+...+a_k}{a_1+a_2+...+a_n}\right)\frac{a_k}{a_1+a_2+...+a_n},$$

and consequently, as $\|\Delta_n\|\to 0$, the limit of this sum will be the $\int_0^1 f(x)dx$.

Example 4. Let $f:[0,1]\to R$ be a Riemann integrable function.
We will prove that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}kf\left(\frac{k^2}{n^2}\right)=\frac{1}{2}\int_0^1 f(x)dx$$

Take the division $\triangle'_n=\left\{0,\frac{1^2}{n^2},\frac{2^2}{n^2},...,\frac{k^2}{n^2},...,1\right\}$ for which the division points are:

$$x_k^n=\frac{k^2}{n^2},\ k\in\{1,2,...,n\}.$$

One can observe that

$$x_k^n-x_{k-1}^n=\frac{2k-1}{n^2}\leq\frac{2n-1}{n^2}$$

and such $\|\triangle'_n\|\to 0$.

Take the intermediate points in the right-end of the subintervals again:

$$\xi_k^n \in \left[x_{k-1}^n, x_k^n\right], \ \xi_k^n = x_k^n = \frac{k^2}{n^2}, \quad k \in \{1, 2, ..., n\};$$

The associated integral sum is:

$$\begin{aligned}\sigma_{\Delta_n'}(f, \xi^n) &= \sum_{k=1}^n f(\xi_k^n)(x_k^n - x_{k-1}^n) = \sum_{k=1}^n \frac{2k-1}{n^2} f\left(\frac{k^2}{n^2}\right) = \\ &= 2\sum_{k=1}^n \frac{k}{n^2} f\left(\frac{k^2}{n^2}\right) - \frac{1}{n^2}\sum_{k=1}^n f\left(\frac{k^2}{n^2}\right).\end{aligned}$$

Consequently:

$$\frac{1}{n^2}\sum_{k=1}^n k f\left(\frac{k^2}{n^2}\right) = \frac{1}{2}\sigma_{\Delta_n'}(f, \xi^n) + \frac{1}{2n^2}\sum_{k=1}^n f\left(\frac{k^2}{n^2}\right) \tag{1}$$

As the function is integrable, and so bounded as well, it must exist an $M > 0$, for which $|f(x)| \leq M$, for all $x \in [0, 1]$. We can conclude

$$\left|\frac{1}{2n^2}\sum_{k=1}^n f\left(\frac{k^2}{n^2}\right)\right| \leq \frac{1}{2n^2}\sum_{k=1}^n \left|f\left(\frac{k^2}{n^2}\right)\right| \leq \frac{nM}{2n^2} \to 0,$$

in other words the last term of expression (1) has limit 0. Taking the limits in equality (1), we have finally the desired result:

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^n k f\left(\frac{k^2}{n^2}\right) = \frac{1}{2}\int_0^1 f(x)dx.$$

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