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Improper integrals and Riemann sums

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Abstract. The aim of this note is to give simple examples of absolutely convergent improper integrals having integral sums whose limit does not equal the value of the integral.

1. Preliminaries

If a function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is Riemann integrable, $a = x_0^{(n)} < x_1^{(n)} < < \ldots < x_{k(n)}^{(n)} = b$ with $\lim\limits_{n\to\infty} \max\left\{x_1^{(n)} - x_0^{(n)}, x_2^{(n)} - x_1^{(n)}, \ldots, x_{k(n)}^{(n)} - x_{k(n)-1}^{(n)}\right\} = 0$ and $\xi_i^{(n)} \in \left[x_{i-1}^{(n)}, x_i^{(n)}\right]$, $\forall i \in \{1, 2, \ldots, k(n)\}, \forall n \in \mathbb{N}^*$, then

$$\lim_{n\to\infty} \sum_{i=1}^{k(n)} \left(x_i^{(n)} - x_{i-1}^{(n)}\right) f\left(\xi_i^{(n)}\right) = \int_a^b f(x)\, dx \qquad (1)$$

holds, i.e. the limit of the integral sums (Riemann sums) equals the value of the integral.

In case of equidistant divisions and $\xi_i^{(n)} \in \left[a + (i-1) \cdot \frac{b-a}{n}, a + \ i \cdot \frac{b-a}{n}\right]$, $i \in \{1, 2, \ldots n\}$, $n \in \mathbb{N}^*$, this is $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{b-a}{n} f\left(\xi_i^{(n)}\right) = \int\limits_a^b f(x)\, dx$.

These formulas allow us to approximate the value of the integral by particular integral sums, as well as to compute the above limits by means of integration.

Transposed to convergent improper integrals these formulas do not always hold. In the paper we build up simple examples to illustrate this fact, moreover, for absolutely convergent improper integrals. Focusing on equidistant divisions of the integration domain will be sufficient for our purpose.

First we recall the following definition.

Definition 1. Consider$f : [a, b) \to \mathbb{R}$, where $a \in \mathbb{R}$, $b \in (a, \infty]$. If the function f is Riemann integrable on every interval $[a, t] \subset [a, b)$ $\quad (t > a, t \in \mathbb{R})$ and $\exists \lim\limits_{t\to b, t<b} \int\limits_a^t f(x)\, dx$, then we say that this limit is the improper integral

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on $[a,b)$ of the function f and we denote it by $\int\limits_{[a,b)} f(x)dx = \lim\limits_{t\to b, t<b} \int\limits_a^t f(x)\,dx$. We say that the improper integral $\int\limits_{[a,b)} f(x)dx$ is convergent, if $\int\limits_{[a,b)} f(x)dx \in \mathbb{R}$ and it is absolutely convergent, if $\int\limits_{[a,b)} |f(x)|\,dx$ is convergent. These notions are of interest if either the domain $[a,b)$, or the function f in the vicinity of b is unbounded. One may consider the similar notions over $(a,b]$.

We also remind that the absolute convergence implies convergence, but not conversely.

Now we prepare some very simple examples of absolutely convergent improper integrals with integral sums whose limit differs from the value of the integral. The first example is integral of unbounded function over bounded integration domain. The second one is integral over unbounded domain.

2. The examples

Example 1. Denote $A = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$. Consider $p > 1$ fixed and

$$f : (0,1] \to \mathbb{R}, f(x) = \begin{cases} 0, & x \in (0,1]\backslash A \\ n^p, & x = \dfrac{1}{n} \in A \end{cases}.$$

On every interval $[t,1] \subset (0,1]$ this function differs from the constant 0 function only in a finite number of points, namely in $x = \frac{1}{n}$ with $n \in \left\{1, 2, 3, \dots, \left[\frac{1}{t}\right]\right\}$ where $[\,]$ means integer part. So on every interval $[t,1] \subset (0,1]$ the function$f$ is Riemann integrable and $\int\limits_t^1 f(x)\,dx = \int\limits_t^1 0dx = 0$.

It results that $\int\limits_{(0,1]} f(x)\,dx = \lim\limits_{t\to 0, t>0} \int\limits_t^1 f(x)\,dx = 0$, so the improper integral $\int\limits_{(0,1]} f(x)\,dx$ is convergent, and since f is nonnegative, it is absolutely convergent.

We examine the sequence of integral sums $\sum\limits_{i=1}^n \frac{1}{n} f\left(i \cdot \frac{1}{n}\right)$, $n \in N^*$. By comparing the sum with its first term, since f is nonnegative, we have $\sum\limits_{i=1}^n \frac{1}{n} f\left(i \cdot \frac{1}{n}\right) \geq \frac{1}{n} \cdot f\left(1 \cdot \frac{1}{n}\right) = n^{p-1}$; $p - 1 > 0$, so $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n \frac{1}{n} f\left(i \cdot \frac{1}{n}\right) = \infty$.

Clearly, this limit of integral sums fails to equal the value of the improper integral.

Example 2. Let $p > 0$. Consider a sequence of positive numbers (y_n) minorized by p, and the function

$$f : [1, \infty) \to \mathbb{R}, f(x) = \begin{cases} 0, & x \in [1, \infty) \setminus \{1, 2, 3, 4, ...\} \\ y_n, & x = n \in \{1, 2, 3, 4, \dots\} \end{cases} .$$

On every interval $[1, t] \subset [1, \infty)$ the function f differs from the constant 0 function only in a finite number of points, namely in $x \in \{1, 2, 3, \cdots, [t]\}$. So, on every interval $[1, t]$ the function f is Riemann integrable and $\int\limits_1^t f(x)\,dx = \int\limits_1^t 0dx = 0$.

It results that $\int\limits_{[1,\infty)} f(x)\,dx = \lim\limits_{t\to\infty} \int\limits_1^t f(x)\,dx = 0$, so the improper integral $\int\limits_{[1,\infty)} f(x)\,dx$ is convergent, and since f is nonnegative, it is absolutely convergent.

Nevertheless, the equality $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{\infty} \frac{1}{n} f\left(1 + i \cdot \frac{1}{n}\right) = \int\limits_{[1,\infty)} f(x)dx$ does not hold. To show this, we examine $\sum\limits_{i=1}^{\infty} \frac{1}{n} f\left(1 + i \cdot \frac{1}{n}\right)$ for $n \in \mathbb{N}^*$ fixed: by summing-up the terms corresponding to $i \in I_n = \{jn | j \in \mathbb{N}^*\}$ in this series, since f is nonnegative, we obtain $\sum\limits_{i=1}^{\infty} \frac{1}{n} f\left(1 + i \cdot \frac{1}{n}\right) \geq \sum\limits_{i \in I_n} \frac{1}{n} f\left(1 + i \cdot \frac{1}{n}\right) =$ $= \sum\limits_{j=1}^{\infty} \frac{1}{n} f(1 + j) = \sum\limits_{j=1}^{\infty} \frac{1}{n} y_{j+1} \geq \sum\limits_{j=1}^{\infty} \frac{1}{n} p = \infty$, so $\sum\limits_{i=1}^{\infty} \frac{1}{n} f\left(1 + i \cdot \frac{1}{n}\right) = \infty$.

Choosing the sequence (y_n) majorized, we get f bounded.

3. Final remarks

The reason why the limit of integral sums does not always equal the value of the convergent improper integral is that in case of improper integrals we perform two limit computing processes – one refines the divisions of an integration domain, the other expands this integration domain – and these processes do not commute unless suitable conditions met.

The following result is immediately.

Proposition 1. *If $a, b \in R, a < b$ and the improper integral $\int\limits_{(a,b]} f(x)\,dx$ is convergent, then $\lim\limits_{t\to a, t>a} \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{b-t}{n} f\left(t + i \cdot \frac{b-t}{n}\right) = \int\limits_{(a,b]} f(x)\,dx$ holds.*

Proof. Apply Definition 1 and then formula (1) for the interval $[t, b]$ by taking $x_i^{(n)} = \xi_i^{(n)} = t + i \cdot \dfrac{b-t}{n}$, $i \in \{1, 2, \ldots, n\}$. $\square$

Remark that the process $n \to \infty$ refines the divisions of the integration domain of the Riemann integral $\int\limits_t^b f(x)\, dx$ and $t \to a, t > a$ expands the interval $[t, b]$ to $(a, b]$.

Under the conditions of Proposition 1, to deduce

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i \cdot \frac{b-a}{n}\right) = \int\limits_{(a,b]} f(x)\, dx \tag{2}$$

we have to invert the two limit computing processes contained in Proposition 1 and to commute $\lim\limits_{t\to a, t>a}$ with the computation of the values of f. These, even if the improper integral is absolutely convergent, may be done only if appropriate conditions are met.

Sufficient conditions which ensure the validity of (2) in special cases of convergent improper integrals, and which can be easily checked practically, one can deduce from [1] or [4].

Sufficient conditions for a function $f : [a, \infty) \to \mathbb{R}$ with convergent improper integral on $[a, \infty)$ to verify equality

$$\lim_{h\to 0, h>0} \sum_{i=1}^{\infty} h f(a + i \cdot h) = \int\limits_{[a,\infty)} f(x)\, dx \tag{3}$$

one can deduce based on [1], [3] or [4].

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