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Some geometric transformations and their connections to complex numbers

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Abstract. In the twelve form textbooks ([2], [3]) are presented some groups of geometric transformations: homotheties, translations, rotations etc.

In this paper we will present two geometric transformations and we will show their relationship to groups of complex functions of the same type, namely isomorphic structures: the group of the rotations of a polygon with n vertices and Klein's group.

At the end, we will show that the type of the group of the rotations of a regular polygon with four vertices (the square) and Klein's group of geometric transformations exhaust all types of groups of order 4.

1. The group of the rotations of a regular polygon with n vertices

1. Let it be the plane π refereed to a cartesian coordinates xOy and $P_0P_1 \dots P_{n-1}$ a regular polygon with n vertices, $n \geq 3$. To simplify the exposition, we consider O the center of the circle circumscribed in the polygon and the radius 1. We define ρ_k, $k = \{0, 1, \dots, n-1\}$ the rotation of center O and oriented angle $\dfrac{2k\pi}{n}$ which makes that every vertex M correspond with a vertex N, so that $m(\widehat{MON}) = \dfrac{2k\pi}{n}$.
Let be $R_n = \{\rho_0, \dots, \rho_{n-1}\}$

Proposition 1. *The set $(R_n, \circ)$, where $\circ$ is the composition of the rotation, is an abelian group, named the group of the rotations of a regular polygon with n vertices.*

Proof. a) $\rho_k \circ \rho_l = \rho_m \in R_n$, where m is the rest from the division of $k + l$ to n.

b) ρ_0 is the neutral element.

c) the associativity of composition of functions is true.

d) for all ρ_k there exists $\rho_k^{-1} = \rho_{n-k} \in R_n$, because $\rho_k \circ \rho_{n-k} = \rho_\circ$.

e) $\rho_k \circ \rho_l = \rho_l \circ \rho_k$, because for $l + k$ and $k + l$ we obtain the same rest dividing them to m. □

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2) Let us, consider, now the set of the norder roots of the unit: $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$, $n \in \mathbb{N}^*$.
It is known that:

Proposition 2. $(U_n, \cdot)$ *is an abelian group.*

Indeed:
a) for all $z_1, z_2 \in U_n$ we have $z_1^n = z_2^n = 1$.
From here $(z_1 z_2)^n = z_1^n \cdot z_2^n = 1$ and so $z_1 z_2 \in U_n$.
b) The multiplication is associative on U_n because he is associative on $\mathbb{C}$.
c) $1 \in U_n$ is the neutral element.
d) The symmetrical of $z \in U_n$ is $\dfrac{1}{z} \in U_n$.
e) The multiplication is commutative.
Using the trigonometrical form of complex numbers we have that:

$$U_n = \left\{ \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \mid k \in \{0, 1, \ldots, n-1\} \right\}$$

Noting $\varepsilon = \cos \dfrac{2\pi}{n} + i \sin \dfrac{2\pi}{n}$ we have $\varepsilon^k = \cos \dfrac{2k\pi}{n} + i \sin \dfrac{2k\pi}{n}$.
Because $\varepsilon^n = 1$, we have $\varepsilon^k = \varepsilon^r$ where r is the rest from the division of k to n. From here we have that:

$$U_n = \{\varepsilon^k \mid k \in \mathbb{Z}\},$$

and because of this it is said that U_n is a cyclical group, namely is generated by ε.

3) Let's show, now, that the two groups are of the same type (isomorphs). We consider the function $\varphi : R_n \to U_n$, $\varphi(\rho_k) = \varepsilon^k$. We have $\varphi(\rho_k \circ \rho_l) = \varphi(\rho_m)$, where m is the rest from the division of $k + l$ to n. On the other hand, $\varepsilon^k \cdot \varepsilon^l = \varepsilon^{k+l} = \varepsilon^m$, where m is the rest from the division of $k + l$ to n. From here we obtain

$$\varphi(\rho_k \circ \rho_l) = \varphi(\rho_m) = \varepsilon^m = \varepsilon^{k+l} = \varepsilon^k \cdot \varepsilon^l = \varphi(\rho_k) \cdot \varphi(\rho_l),$$

namely φ is a morphism of groups.
Obviously φ is a bijection. From here, we deduce that the two groups are of the same type (isomorphs). We can also say that these groups coincide except of an isomorphism.

2. Klein's group

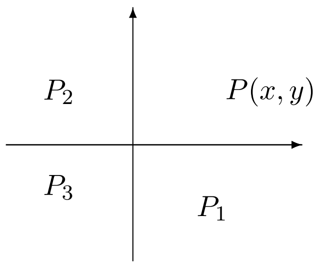
1) Let π be the plane refereed to the reper xOy. There are defined the following transformations:
$1_\pi : \pi \to \pi$, $1_\pi\left(P(x, y)\right) = P(x, y)$ - the identical transformation
$s_x : \pi \to \pi$, $s_x\left(P(x, y)\right) = P_1(x, -y)$ - the reflection with respect to Ox line

$s_y : \pi \to \pi,\ s_y\,(P(x,y)) = P_2(-x,y)$ - the reflection with respect to Oy line

$s_0 : \pi \to \pi,\ s_0\,(P(x,y)) = P_3(-x,-y)$ - the reflection with respect to O.

On the set $K = \{1_\pi, s_x, s_y, s_0\}$ the operation of composition is given by the following table:

0	1_π	s_x	s_y	s_0
1_π	1_π	s_x	s_y	s_0
s_x	s_x	1_π	s_0	s_y
s_y	s_y	s_0	1_π	s_x
s_0	s_0	s_y	s_x	1_π



From this table we obtain that the operation of composition is closed and that every $f \in K$ implies $f^{-1} \in K$ and $f^{-1} = f$.

Also it results that the operation of composition is commutative. From here $(K, \circ)$ is a commutative group, named Klein's group.

2) Consider, now, the set $F_k = \{f_0, f_1, f_2, f_3\}$ where:

$$\begin{aligned} f_0 &: \mathbb{C} \to \mathbb{C}, \quad f_0(z) = z \\ f_1 &: \mathbb{C} \to \mathbb{C}, \quad f_1(z) = \overline{z} \\ f_2 &: \mathbb{C} \to \mathbb{C}, \quad f_2(z) = -\overline{z} \\ f_3 &: C \to \mathbb{C}, \quad f_3(z) = -z \end{aligned}$$

On the set F_k the operation of composition of the functions is given by the following table:

0	f_0	f_1	f_2	f_3
f_0	f_0	f_1	f_2	f_3
f_1	f_1	f_0	f_3	f_2
f_2	f_2	f_3	f_0	f_1
f_3	f_3	f_2	f_1	f_0

From the table we have that the operation of composition is closed and that every $f \in F_k$ implies $f^{-1} \in F_k$ and $f^{-1} = f$. So $(F_k, \circ)$ is a abelian group.

3) Let's show,now, that the groups $(K, \circ)$ and $(F_k, \circ)$ are of the same type.

Let it be $\varphi : K \to F_k$, given by $\varphi(1_\pi) = f_0$, $\varphi(s_x) = f_1$, $\varphi(s_y) = f_2$ and $\varphi(s_0) = f_3$. It is obvious that φ is a bijection.

We also have $\varphi(s_x \circ s_y) = \varphi(s_0) = f_3 = f_1 \circ f_2 = \varphi(s_x) \circ \varphi(s_y)$ and the others.

Otherwise we notice that the two tables superpose identical through the correspondence φ. We proved in this way

Proposition 3. *The groups $(K, \circ)$ and $(F_k, \circ)$ are of the same type or identical, except of an isomorphism.*

3. Groups of order 4

Consider a Group $(G, \cdot)$ of order 4. If there exists an element $a \in G$ of order 4, then the order of the group $\langle a \rangle$ is 4 and from here it results that $G = \langle a \rangle$, namely G is cyclical.

If there doesn't exist $a \in G$ with the order equal with 4, then $a \in G$, $a \neq e$, we have that the order of a is 2. Hence $x^2 = 1$, for all $x \in G$ and so $(G, \cdot)$ is abelian group.

Indeed, we have that:

$$x \cdot y = x \cdot e \cdot y = x(xy)^2 \cdot y = x^2 \cdot yx \cdot y^2 = e \cdot yx \cdot e = y \cdot x$$

Moreover, if $a \in G$, $a \neq e$ and $H = \langle a \rangle = \{e, a\}$ we have that the order of G/H is 2. Choosing an element $b \in G \backslash H$, we have $G = H \cup H_b = \{e, a, b, ab\}$. So, the group G is defined by the generators a and b and the relations $a^2 = e$ and $b^2 = e$, $ab = ba$, and its table is:

$\cdot$	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

From the things said above, it results that we have two types of 4 order groups: 4 order cyclical group and the one that is not cyclical and which is proofed with the help of the above table of multiplication.

In conclusion, every 4 order groups is cyclical, of the type U_4, namely isomorph with U_4 or with the group of the rotations of a regular polygon with four vertices (the square), or is the type K, namely isomorph with Klein's groupe (the group of geometric transformations).

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