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Quadrilaterals in which an angle is the mean of the other angles

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Abstract. In this article we determine all the quadrilaterals in which an angle is the arithmetic or geometric mean of the other angles. The angles are measured in degrees. We also study the particular case where the measures of the angles are integers numbers. In the sequel, is studied the same problems for all types of means, in the special case of inscribable quadrilaterals.

In [2] L. Trouché determine the triangles in which an angle is the geometric mean of the other angles. Also, he proves that from all the triangles only equilateral triangles have measures of the angles, integer numbers.

In [1] we generalize this problem for the harmonic and quadratic mean.
We proved that the set of $(x, y) \in (0°, 180°) \times (0°, 180°)$, that is $x + y + m(x, y) = 180°$, where $m(x, y)$ is this means, are an arc of two hyperboles, and also only equilateral triangles have the properties that the measures of their angles are integers numbers.

1. The case of any quadrilaterals

In this article we determine an analogous problem for quadrilaterals.

Let x, y, z and t be the measures in degrees of the angles in a quadrilateral, we suppose that t is the mean of x, y, z that is $t = m(x, y, z)$. Obviously $x, y, z, t \neq 180°$.

1°. *An angle is the arithmetic mean of the others*, that is $t = \dfrac{x + y + z}{3}$. Because $x + y + z + t = 360°$, we have $t = 90°$ and $x + y + z = 270°$, where $x, y, z \in (0°, 270°)\backslash\{180°\}$.

Let $I = (0, 270°)\backslash\{180°\}$, therefore the set $\{(x, y, z) \in I^3\,; x+y+z = 270°\}$ is the interior of the triangle ABC from the figure 1, without the segments MN, PQ, SR.

Notice that not only the quadrilaterals with the angles $(x, y, 270° - x - y, 90°)$ are different from those with the angles given by the sets $(y, x, 270 - x - y, 90°)$ and $(y, 90°, x, 270° - x - y)$ and so on.

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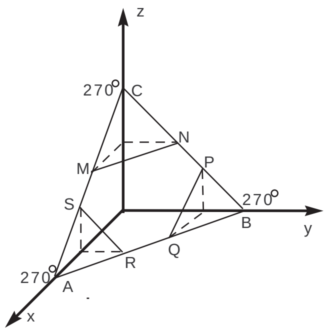

FIGURE 1

Even for the same angles there are many classes of quadrilaterals, for example both squares and rectangles have the same angles $(90°, 90°, 90°, 90°)$.

In the sequel, we consider that $0 < x \leq y \leq z < 360°$; $x, y, z \neq 180°$ and $t = m(x, y, z)$, everywhere.

$2°$. *An angle is the geometric mean of the others.*
If $t = \sqrt[3]{xyz}$, then we have

$$x + y + z + \sqrt[3]{xyz} = 360° \tag{1.1}$$

and because $\sqrt[3]{xyz} \leq \dfrac{x+y+z}{3} = \dfrac{360° - \sqrt[3]{xyz}}{3}$, we have $\sqrt[3]{xyz} \leq 90°$. The locus of the set $\{(x, y, z)$ where (1.1) is satisfied $\}$ is part of a surface.

If we consider the measures x, y, z in integers degrees, and $0 < x \leq y \leq z$, let d be the greatest common divisor of x, y, z and $x = dx'$, $y = dy'$, $z = dz'$, so that $(x', y', z') = 1$ are relatively prime.
From (1.1) we have that d divide 360 and

$$x' + y' + z' + \sqrt[3]{x'y'z'} = D \tag{1.2}$$

where $D = \dfrac{360°}{d}$ is a divisor of 360.

We distinguish the following cases:

a) if $x' = 1$; $y' = b$; $z' = b^2$ and $\sqrt[3]{x'y'z'} = b$, then $1 + 2b + b^2 = D$, therefore $(b+1)^2 = D$ where $D \in \{4, 9, 36\}$, $b = 1$ or 2 respectively 5 and $d = 90$ or 40 respectively 10.

For $d = 90$ and $b = 1$ we have quadrilaterals with the angles $(90, 90, 90, 90)$.

For $d = 40$ and $b = 2$ we have quadrilaterals with the angles $(40, 80, 160, 80)$, in any order.

For $d = 10$ and $b = 5$ we have the quadrilaterals with the angles $(10, 50, 250, 50)$ in any order.

b) If $x' = 1$; $y' = 1$; $z' = c^3$; $c \geq 1$ we have $\sqrt[3]{x'y'z'} = c$ and $2 + c + c^3 = D$ where D is a divisor of $360°$, $D \geq 4$.

So, for $c = 2$ we have $D = 12$ and $d = 30$, $x = 30$, $y = 30$; $z = 240$ and $\sqrt[3]{xyz} = 60$ and we obtain $(30, 30, 240, 60)$ in any order.

c) If $x' = 1$; $y' = b^2$; $z' = bc^3$, where $b \leq c^3$ we have $\sqrt[3]{x'y'z'} = bc$ and $1 + b^2 + bc^3 + bc = D$ or $b(b + c^2 + c) = D - 1$, where D divide $360°$. For $b \geq 2$ is necessary that $D \geq 30$. For $b = 2$ and $c = 2$ we have $2(2 + 4 + 2) = 16$ and $D = 17$ that is impossible.

For $b = 3$ and $c = 2$ we have $x' = 1$; $y' = 9$; $z' = 24$ and $\sqrt[3]{x'y'z'} = 6$ and therefore $D = 40$ and $d = 9$. For that $x = 9$; $y = 81$; $z = 216$ and $\sqrt[3]{xyz} = 54$ and we have $(9, 81, 216, 54)$ and, of course, any permutation of this.

For $b = 7$ and $c = 2$ we have $x' = 1$; $y' = 49$; $z' = 56$ and $\sqrt[3]{x'y'z'} = 14$, therefore $D = 120$ and $d = 3$. For that we have $x = 3$, $y = 147$, $z = 168$ and $\sqrt[3]{xyz} = 42$, so the solution is $(3, 147, 168, 42)$ and any permutation of this.

d) Now let $x' = a$; $y' = a^2$; $z' = b^3$; $(0, b) = 1$, $a \geq 2$; $b \geq 2$ and $a^2 \leq b$ since $\sqrt[3]{x'y'z'} = ab$, we have $a + a^2 + b^3 + ab = b$ where $D \geq 18$. In fact, only $a = 5$ and $b = 3$ for which $D = 72$ and $d = 5$ to get to a solution $x' = 5$; $y' = 25$; $z' = 27$; $\sqrt[3]{x'y'z'} = 15$. From this we have $x = 25$, $y = 125$; $z = 135$ and $\sqrt[3]{xyz} = 75$ we have in this way the solution $(25, 125, 135, 75)$ and any permutation of this.

It is possible to study other combination such that $\sqrt{x'y'z'} \in \mathbb{N}^*$; $(x', y', z') = 1$ and (1.2) are satisfied, as:

$$x' = a;\ y' = a^2 b;\ z' = b^3;\ (a, b) = 1$$

$$x' = a^3;\ y' = b^3;\ z' = c^3;\ (a, b, c) = 1$$

$$x' = a^2 b;\ y' = ab^2;\ z' = c^3 = (a, b, c) = 1$$

and so on.

But we deduce that there are only the above solutions.

We also can use the following C program to compute the solution for the equation (1.1):

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main(){
long x,y,z;
clrscr();
printf(" x | y | z ");
for(x=1;x<360;x++)
  for(y=1;y<360;y++)
```

```
    for(z=1;z<360;z++){
    long val1;
    long val2;
    if((x<=y)&&(y<=z))
      if((pow (val1,3)==val2)&&(val1>0)){
        printf(" %5ld |%5ld|%5ld",x,y,z);
      }
    }
  }
```

So, we have the following cases:

x	y	z	$\sqrt[3]{x*y*z}$
3	147	168	42
9	81	216	54
10	50	250	50
25	125	135	75
30	30	240	60
40	80	160	80
90	90	90	90

2. The case of inscribable quadrilateral

Let $0 < x \leq y \leq 90^\circ$ be the measures of sharps angles of an inscribable quadrilateral and $90 \leq 180 - y \leq 180 - x < 180$ the others angles.

1°. As by general case, *if an angle is the arithmetic mean of others*, then his measure is 90 degrees. Because the opposite angles are supplementary, the propose set of inscribable quadrilaterals is

$$\left\{ABCD;\ m\widehat{A} = m\widehat{C} = 90^\circ;\ m\widehat{D} = 180^\circ - m\widehat{B}\right\}$$

Among all this, there are ninety classes of inscribable quadrilaterals, for which the angles measures in integers numbers degrees:

$$(k, 90^\circ, 180^\circ - k, 90^\circ)\,; k = \overline{1, 90}.$$

2°. *If an angle is the geometric mean of the others*, then his measure is less from 90° and we have the equation

$$y = \sqrt[3]{x(180 - x)(180 - y)}$$

or

$$\frac{y^3}{180 - y} = x(180 - x) \tag{2.1}$$

If $y = 90 - a$; $x = 90 - b$ where $0 \le a \le b < 90°$ we obtain

$$b^2 = 90^2 - \frac{(90-a)^3}{90+a} \tag{2.2}$$

which is an arc of a semicubic parabola.

To result from general case I.1 that only the rectangles (and squares) have the measures of the angles to express in integers numbers.

3°. *If an angle of inscribable quadrilaterals is the harmonic mean of other*, then it is y and

$$\frac{3}{y} = \frac{1}{x} + \frac{1}{180-x} + \frac{1}{180-y} \tag{2.3}$$

where $0 < x \le y \le 90$, or

$$\frac{y(180-y)}{135-y} = \frac{x(180-x)}{45}$$

If $a = 90 - y$; $b = 90 - x$; $0 \le a \le b < 90$, then we obtain

$$b^2 = 45\left(a + 135 - \frac{3 \cdot 45^2}{a+45}\right) \tag{2.4}$$

From condition $b^2 \in \mathbb{N}$ we have $a = 5k$, where $k = \overline{0,17}$, therefore

$$b^2 = 15^2\left(k + 27 - \frac{243}{k+9}\right) \tag{2.5}$$

Because $k + 9$ is an divisor of 243 and $k < 18$, is possible only $k = 0$, that is $a = 0$ and $b = 0$. Consequently $x = y = 90°$. To result again that only rectangles have an angle (measured in the degrees), the harmonic mean of others.

The following C program verify that.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
long x,y;
clrscr();
printf(" x | y ");
  for(x=1;x<90;x++)
    for(y=1;y<360;y++)

    if((x<=y)
      if(3*x*(180-x)*(180-y)==y*((180-x)*(180-y)+
        +x*(180-x)+x*(180-y)))
```

```
            printf(" %5ld |%5ld",x,y);
          }
       }
   }
```

4°. *If an angle of invertible quadrilaterals is the quadratic mean of the others*, then is bigger from 90. With the above notations, let $180-y$ be this angle.

We have

$$\sqrt{\frac{x^2+y^2+(180-x)^2}{3}} = 180-y$$

or $x^2-y^2-180x+540y-180^2=0$.

This represent an arc of a equilateral hyperbola (see Figure 2):

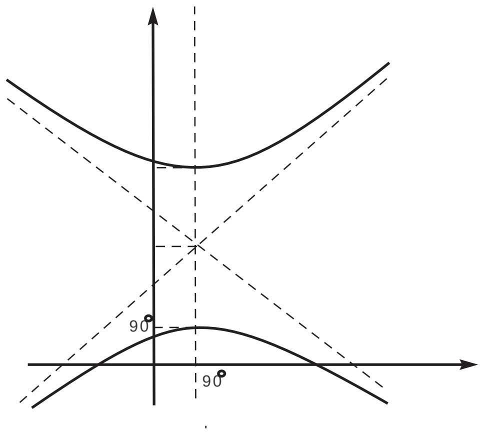

FIGURE 2

$$(270-y)^2-(90-x)^2=180^2 \tag{2.6}$$

The solution in integers numbers it obtain writing (2.7) as:

$$(360-x-y)(180+x-y)=180^2 \tag{2.7}$$

where $180+x-y \leq 360-x-y$.

Let $d=180+x-y$ be an natural divisor of 180^2; $d \leq 180$.

From $\begin{cases} 360-x-y=\dfrac{180^2}{d} \\ 180+x-y=d \end{cases}$ we have

$$\begin{cases} x = 90+\dfrac{d}{2}-\dfrac{180^2}{2d} \\ y = 270-\dfrac{d}{2}-\dfrac{180^2}{2d} \end{cases} \tag{2.8}$$

From $x, y \in \mathbb{N}^*$; $0 < x \leq y \leq 90$, we have $d \in \left(90\left(\sqrt{5}-1\right); 180\right]$. But d is an even divisor of $180^2 = 2^4 \cdot 3^4 \cdot 5^2$, therefore $d = 2^\alpha \cdot 3^\beta \cdot 5^\gamma$, where $1 \leq \alpha \leq 4$; $0 \leq \beta \leq 4$; $0 \leq \gamma \leq 2$. We distinguish the following cases:

d	x	y	The measures of angles of inscribable quadrilateral
120	15	75	$(15°, 75°, 165°, 105°)$
150	57	87	$(57°, 87°, 123°, 103°)$
162	71	89	$(71°, 89°, 109°, 101°)$
180	90	90	$(90°, 90°, 90°, 90°)$

We also, resolved the above equation using the following C program

```
#include <stdio.h>
#include <conio.h>
#include <math.h>
void main ()
{
indent int i,j;
  for(i=0;i<=90;i++)
    for(j=0;j<=90;j++)
    {
      if(pow(270-i,2)-pow(90-j,2)==180*180){
        printf(" %d | %d ",i,j);
      }

    }
}
```

References

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