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Some remarks on orthogonal polynomials

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Abstract. In this article we demonstrate some general results, from which, through particularities, we obtain identities satisfied by Legendre's, Laguerre's and Hemite's polynomials.

For $k \in \mathbb{N}$, let $\mathbb{P}_k$ be the set of the polynomials of real coefficients, with maximum degree k.

Theorem 1. *Let the weight function $w : [a,b] \to \mathbb{R}^*$, $w \in C[a,b]$ and $k \in \mathbb{N}$. The function $\varphi_{k+1} : [a,b] \to \mathbb{R}$, $\varphi_{k+1} \in C[a,b]$ satisfies in report with the function w the orthogonality condition*

$$\int_a^b w(x)\varphi_{k+1}(x)p(x)dx = 0, \quad \forall\, p \in \mathbb{P}_k \tag{1}$$

if and only if there exists a function $u : [a,b] \to \mathbb{R}$, $k+1$ times derivable on $[a,b]$, which verifies the conditions

$$u^{(k+1)}(x) = w(x)\varphi_{k+1}(x), \quad \forall\, x \in [a,b] \tag{2}$$

and

$$u^{(i)}(a) = u^{(i)}(b) = 0, \quad \forall\, i \in \{0,1,\ldots,k\}. \tag{3}$$

Proof. It can be found in [1] or [3]. □

Theorem 2. *Let the weight function $w : [a,b] \to \mathbb{R}_+^*$, $w \in C[a,b]$, $n \in \mathbb{N}$, u_n, $\varphi_n : [a,b] \to \mathbb{R}$, u_n, n times derivable on $[a,b]$, $\varphi_n \in C[a,b]$, so*

$$\int_a^b w(x)\varphi_n(x)p(x)dx = 0\,, \quad \forall\, p \in \mathbb{P}_{n-1}\,, \tag{4}$$

$$u_n^{(n)}(x) = w(x)\varphi_n(x)\,, \quad \forall\, x \in [a,b]\,, \tag{5}$$

$$u_n^{(i)}(a) = u_n^{(i)}(b) = 0\,, \quad \forall\, i \in \{0,1,\ldots,n-1\} \tag{6}$$

and

$$\varphi_n \in \mathbb{P}_n,\ \varphi_n\,(x) = a_n x^n + b_n x^{n-1} + \cdots\,, \quad a_n \neq 0\,. \tag{7}$$

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Then

$$\int_a^b w(x)\,\varphi_n(x)\,\varphi_m(x)dx = \begin{cases} 0, & m \neq n \\ (-1)^n\, n!\, a_n \int_a^b u_n(x)dx, & m = n \end{cases} \tag{8}$$

and

$$\int_a^b w(x)\,\widetilde{\varphi}_n(x)\,\widetilde{\varphi}_m(x)dx = \begin{cases} 0, & m \neq n \\ \dfrac{(-1)^n\, n!\, \int_a^b u_n(x)dx}{a_n}, & m = n\,. \end{cases} \tag{9}$$

Proof. If $m \neq n$, taking (4) and (7) into account it results that $\int_a^b w(x)\,\varphi_n(x)\,\varphi_m(x)dx = 0$. If $m = n$ we have $\|\varphi_n\|^2 = \int_a^b w(x)\,\varphi_n^2(x)dx = = \int_a^b w(x)\,\varphi_n(x)\,\varphi_n(x)dx$ and taking (5) into account we obtain $\|\varphi_n\|^2 = = \int_a^b u_n^{(n)}(x)\,\varphi_n(x)dx$. Integrating through parts and considering (6), we have

$$\begin{aligned} \|\varphi_n\|^2 &= u_n^{(n-1)}(x)\,\varphi_n(x)\Big|_a^b - \int_a^b u_n^{(n-1)}(x)\,\varphi_n'(x)dx = \\ &= -\int_a^b u_n^{(n-1)}(x)\,\varphi_n'(x)dx = \cdots = (-1)^n \int_a^b u_n(x)\,\varphi_n^{(n)}(x)dx\,. \end{aligned}$$

According to (7), we obtain $\|\varphi_n\|^2 = (-1)^n\, n!\, a_n \int_a^b u_n(x)dx$, so (8) takes place. Because $\widetilde{\varphi}_n(x) = \dfrac{1}{a_n}\varphi_n(x)$, from (8) we obtain (9). □

Theorem 3. *Let $\{p_n, n \in \mathbb{N}\}$ be a set of orthogonal polynomials and $p_n(x) = a_n x^n + b_n x^{n-1} + \cdots,\ a_n \neq 0$. Then*

$$\begin{aligned} &\frac{a_n}{a_{n+1}}\, p_{n+1}(x) = \\ &= \left(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} + x\right) p_n(x) - \frac{a_{n-1}}{a_n}\left(\frac{\|p_n\|}{\|p_{n-1}\|}\right)^2 p_{n-1}(x) \end{aligned} \tag{10}$$

and

$$\widetilde{p}_{n+1}(x) = \left(b'_{n+1} - b'_n + x\right)\widetilde{p}_n(x) - \left(\frac{\|\widetilde{p}_n\|}{\|\widetilde{p}_{n-1}\|}\right)^2 \widetilde{p}_{n-1}(x)\,, \tag{11}$$

where $b'_n = \dfrac{b_n}{a_n}$.

Proof. It can be found in [1] or [3]. □

Theorem 4. *In the conditions of Theorem 1 we have that*

$$\frac{a_n}{a_{n+1}}\,\varphi_{n+1}(x) = \left(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} + x\right)\varphi_n(x) + n\,\frac{\int_a^b u_n(x)dx}{\int_a^b u_{n-1}(x)dx}\,\varphi_{n-1}(x) \tag{12}$$

and

$$\widetilde{\varphi}_{n+1}(x) = \big(b'_{n+1} - b'_n + x\big)\widetilde{\varphi}_n(x) + n\,\frac{a_{n-1}}{a_n}\,\frac{\int_a^b u_n(x)dx}{\int_a^b u_{n-1}(x)dx}\;\widetilde{\varphi}_{n-1}(x) \tag{13}$$

Proof. We use Theorem 2 and Theorem 3. □

Next, by using Theorem 2 and Theorem 4, we will give some applications.

Application 1. We consider $w, u_n, \varphi_n : [-1,1] \to \mathbb{R}$, $a = -1$, $b = 1$, $w(x) = 1$, $u_n(x) = (x^2-1)^n$, $\varphi_n(x) = \big[(x^2-1)^n\big]^{(n)}$, so we obtain Legendre's polynomials. Because $\varphi_n(x) = \dfrac{(2n)!}{n!}x^n + 0\cdot x^{n-1} + \cdots$ and $\int_{-1}^1 u_n(x)dx =$ $= (-1)^n\,\dfrac{(n!)^2}{(2n+1)!}\,2^{2n+1}$, then $\widetilde{l_n}(x) = \dfrac{n!}{(2n)!}\varphi_n(x)$ and defining $l_n(x) =$ $= \varphi_n(x)$, we have

$$\int_{-1}^{1} l_n(x)\,l_m(x)dx = \begin{cases} 0, & m \neq n \\ \dfrac{(n!)^2}{2n+1}\,2^{2n+1}, & m = n\,, \end{cases} \tag{14}$$

$$\int_{-1}^{1} \widetilde{l}_n(x)\,\widetilde{l}_m(x)dx = \begin{cases} 0, & m \neq n \\ \dfrac{(n!)^4\cdot 2^{2n+1}}{\big[(2n)!\big]^2(2n+1)}, & m = n\,, \end{cases} \tag{15}$$

$$l_{n+1}(x) = 2(2n+1)x\,l_n(x) - 4n^2 l_{n-1}(x) \tag{16}$$

and

$$\widetilde{l}_{n+1}(x) = x\,\widetilde{l}_n(x) - \frac{n^2}{(2n-1)(2n+1)}\,\widetilde{l}_{n-1}(x)\,. \tag{17}$$

Application 2. We consider $w(x) = e^{-x}$, $u_n(x) = e^{-x}\,x^n$, $a = 0$, $b = \infty$. Then $\varphi_n(x) = g_n(x) = e^x\,(e^{-x}\,x^n)^{(n)}$, so we obtain Laguerre's polynomials. Because $g_n(x) = (-1)^n\,x^n + (-1)^{n-1}\,nx^{n-1} + \ldots$ and $\int\limits_0^\infty u_n(x)dx = n!$, then $\widetilde{g}_n(x) = (-1)^n g_n(x)$ and

$$\int_0^\infty e^{-x} g_n(x) g_m(x)dx = \begin{cases} 0, & m \neq n \\ (n!)^2, & m = n \end{cases}\,, \tag{18}$$

$$\int_0^\infty e^{-x} \widetilde{g}_n(x) \widetilde{g}_m(x)dx = \begin{cases} 0, & m \neq n \\ (n!)^2, & m = n \end{cases}\,, \tag{19}$$

$$g_{n+1}(x) = (2n+1-x)g_n(x) - n^2 g_{n-1}(x) \tag{20}$$

and

$$\widetilde{g}_{n+1}(x)=(x-2n-1)\widetilde{g}_n(x)-n^2\widetilde{g}_{n-1}(x)\,. \tag{21}$$

Application 3. We consider $w(x)=e^{-x^2}$, $u_n(x)=u(x)=e^{-x^2}$, $h_n^*(x)=u^{(n)}(x)e^{x^2}$, $a=-\infty$, $b=\infty$, so we obtain a Hermite type polynomials, $h_n(x)=(-1)^n h_n^*(x)$. Because $h_n^*(x)=e^{x^2}\left(e^{-x^2}\right)^{(n)}=(-1)^n 2^n x^n+$ $+0\cdot x^{n-1}+\cdots$ and $\int\limits_{-\infty}^{\infty} u_n(x)dx=\sqrt{\pi}$, then $\widetilde{h}_n^*(x)=\dfrac{1}{(-1)^n\,2^n}\,h_n^*(x)$ and

$$\int_{-\infty}^{\infty} e^{-x^2}h_n^*(x)h_m^*(x)dx=\begin{cases}0, & m\neq n\\ 2^n n!\sqrt{\pi}, & m=n\end{cases}, \tag{22}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\widetilde{h}_n^*(x)\widetilde{h}_m^*(x)dx=\begin{cases}0, m\neq n & \\ \dfrac{n!\sqrt{\pi}}{2^n}, & m=n\end{cases}, \tag{23}$$

$$\int_{-\infty}^{\infty} e^{-x^2}h_n(x)h_m(x)dx=\begin{cases}0, & m\neq 0\\ 2^n n!\sqrt{\pi}\,, & m=n\end{cases}, \tag{24}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\widetilde{h}_n(x)\widetilde{h}_m(x)dx=\begin{cases}0, & m\neq n\\ \dfrac{n!\sqrt{\pi}}{2^n}, & m=n\end{cases}, \tag{25}$$

$$h_{n+1}^*(x)=-2x\,h_n^*(x)-2n\,h_{n-1}^*(x)\,, \tag{26}$$

$$\widetilde{h}_{n+1}^*(x)=x\,\widetilde{h}_n^*(x)-\frac{1}{2}\,n\,\widetilde{h}_{n-1}^*(x)\,, \tag{27}$$

$$h_{n+1}(x)=2xh_n(x)-2nh_{n-1}(x) \tag{28}$$

and

$$\widetilde{h}_{n+1}(x)=-x\,\widetilde{h}_n(x)-\frac{1}{2}\,n\,\widetilde{h}_{n-1}(x)\,. \tag{29}$$

The readers can find appropriate applications of the next theorem.

Theorem 5. *Let the weight function be $w:[a,b]\to\mathbb{R}_+^*$, $w\in C[a,b]$, $n\in\mathbb{N}$, $\varphi_n:[a,b]\to\mathbb{R}$, $\varphi_n\in C[a,b]$ and there exists $v:[a,b]\to\mathbb{R}$, $2n$ times derivable on $[a,b]$, so*

$$v^{(i)}(a)=0\,,\quad i\in\{1,2,\ldots,2n-1\}\,, \tag{30}$$

$$v^{(2n)}(x)=w(x)\,,\quad \forall\, x\in[a,b]\,, \tag{31}$$

$$\int_a^b w(x)\varphi_n(x)p(x)dx=0\,,\quad \forall\, p\in\mathbb{P}_{n-1} \tag{32}$$

and

$$\varphi_n\in\mathbb{P}_n,\quad \varphi_n\,(x)=a_nx^n+b_nx^{n-1}+\cdots\,,\quad a_n\neq 0\,. \tag{33}$$

Let us prove that

$$\int_a^b w(x)\varphi_n(x)\varphi_m(x)dx =$$
$$= \begin{cases} 0, & m \neq n \\ \sum\limits_{k=1}^{2n} (-1)^{k-1} v^{2n-k}(b)\big(\varphi_n^2\big)^{(k-1)}(b) + (2n)!\, a_n^2 \int\limits_a^b v(x)dx, & m = n \end{cases} \tag{34}$$

and

$$\int_a^b w(x)\,\widetilde{\varphi}_n(x)\,\widetilde{\varphi}_m(x)dx =$$
$$= \begin{cases} 0, & m \neq n \\ \dfrac{1}{a_n} \sum\limits_{k=1}^{2n} (-1)^{k-1} v^{(2n-k)}(b)\big(\varphi_n^2\big)^{(k-1)}(b) + (2n)!\, a_n \int\limits_a^b v(x)dx, & m = n\,. \end{cases} \tag{35}$$

Proof. We have that

$$\begin{aligned}
\|\varphi_n\|^2 &= \int_a^b w(x)\varphi_n^2(x)dx = \int_a^b v^{(2n)}(x)\varphi_n^2(x)dx = \\
&= v^{(2n-1)}(x)\big(\varphi_n^2\big)(x)\Big|_a^b - \int_a^b v^{(2n-1)}(x)\big(\varphi_n^2\big)'(x)dx = \\
&= v^{(2n-1)}(b)\big(\varphi_n^2\big)(b) - v^{(2n-2)}(x)\big(\varphi_n^2\big)'(x)\Big|_a^b + \\
&+ \int_a^b v^{(2n-2)}(x)\big(\varphi_n^2\big)''(x)dx = \cdots = \\
&= \sum_{k=1}^{2n} (-1)^{k-1} v^{(2n-k)}(b)\big(\varphi_n^2\big)^{(k-1)}(b) + \int_a^b v(x)\big(\varphi_n^2\big)^{(2n)}(x)dx = \\
&= \sum_{k=1}^{2n} (-1)^{k-1} v^{(2n-k)}(b)\big(\varphi_n^2\big)^{(k-1)}(b) + (2n)!\, a_n^2 \int_a^b v(x)dx\,,
\end{aligned}$$

the equality from (34) for $m = n$. If $m \neq n$, we use (32) and (33). Because $\widetilde{\varphi}_n(x) = \dfrac{1}{a_n}\varphi_n(x)$, from (34), (35) results. □

Observation. The function v with the properties in Theorem 5 can be obtained, for example, integrating the function w on $[a, x]$ $2n$-times.

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