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On some inequalities for right triangles

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Abstract. We will prove some inequalities between the elements of a right triangle.

In any ABC right triangle we denote $AB = c$, $BC = a$, $CA = b$, and A, B, C be the angles of the triangle, $p = \dfrac{a+b+c}{2}$ the semiperimeter, r, r_a, R the radii of incircle, excircle corresponding to BC and circumcircle of ABC, respectively.

Proposition 1. *In any ABC triangle we have*

$$r = \frac{1}{2}\left(\sqrt{a^2\left(1+\operatorname{tg}^2\frac{A}{2}\right)-(b-c)^2} - a\operatorname{tg}\frac{A}{2}\right) \tag{1}$$

$$\frac{R}{2} = \frac{1}{\sqrt{\sin^2 A\left(1+\operatorname{tg}^2\frac{A}{2}\right)-(\sin B-\sin C)^2} - \sin A\cdot tg\frac{A}{2}} \tag{2}$$

and

$$\frac{R}{r} = \frac{1}{2\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)} \tag{3}$$

Proof. Let I be in centre to an ABC triangle, $IT \perp AC$, $T \in AC$. Because $AT = p - a$, we have $\operatorname{tg}\dfrac{A}{2} = \dfrac{IT}{AT} = \dfrac{r}{p-a}$ and analogous.

From $A = \pi - (B + C)$, results $\dfrac{A}{2} = \dfrac{\pi}{2} - \left(\dfrac{B}{2}+\dfrac{C}{2}\right)$, hence

$$\begin{aligned}\operatorname{tg}\frac{A}{2} &= \operatorname{tg}\left(\frac{\pi}{2}-\left(\frac{B}{2}+\frac{C}{2}\right)\right) = \operatorname{ctg}\left(\frac{B}{2}+\frac{C}{2}\right) = \frac{1}{\operatorname{tg}\left(\frac{B}{2}+\frac{C}{2}\right)} = \\ &= \frac{1-\operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2}}{\operatorname{tg}\frac{B}{2}+\operatorname{tg}\frac{C}{2}} = \frac{1-\frac{r}{p-b}\frac{r}{p-c}}{\frac{r}{p-b}+\frac{r}{p-c}} = \frac{(p-b)(p-c)-r^2}{ar}.\end{aligned}$$

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Thus $r^2+ar\operatorname{tg}\dfrac{A}{2}=(p-b)(p-c)$, is equivalent to $4r^2+4ar\operatorname{tg}\dfrac{A}{2}=a^2-(b-c)^2$, or $\left(2r+a\operatorname{tg}\dfrac{A}{2}\right)^2=a^2\left(1+\operatorname{tg}^2\dfrac{A}{2}\right)-(b-c)^2$. Because $a>|b-c|$, results $a^2\left(1+\operatorname{tg}^2\dfrac{A}{2}\right)-(b-c)^2>0$ and so from the relation above we get (1). Identity (2) can be obtained from (1) by changing sides a,b,c in the sinus theorem. Relation (3) results from (2) after calculations. □

Proposition 2. *In any ABC triangle take place the following inequalities:*

$$r\le\frac{a\left(1-\sin\dfrac{A}{2}\right)}{2\cos\dfrac{A}{2}}\tag{4}$$

$$\frac{R}{r}\ge\frac{1}{2\sin\dfrac{A}{2}\left(1-\sin\dfrac{A}{2}\right)}\tag{5}$$

$$r\le\frac{a}{2}\operatorname{tg}\left(\frac{\pi}{4}-\frac{A}{4}\right)\tag{6}$$

$$\frac{R}{r}\ge\frac{1}{\sin A\cdot\operatorname{tg}\left(\dfrac{\pi}{4}-\dfrac{A}{4}\right)}\tag{7}$$

Equality in any of the inequalities $(4)-(7)$ *can be obtained if and only if* $B=C$.

Proof. From (1) we have $r\le\dfrac{1}{2}\left(\sqrt{a^2\left(1+\operatorname{tg}^2\dfrac{A}{2}\right)}-a\operatorname{tg}\dfrac{A}{2}\right)$ and so we get (4) and then (5). (3) shows us that

$$r\le\frac{a\left(\cos^2\dfrac{A}{4}+\sin^2\dfrac{A}{4}-2\sin\dfrac{A}{4}\cos\dfrac{A}{4}\right)}{2\left(\cos^2\dfrac{A}{4}-\sin^2\dfrac{A}{4}\right)}=\frac{a\left(\cos\dfrac{A}{4}-\sin\dfrac{A}{4}\right)}{2\left(\cos\dfrac{A}{4}+\sin\dfrac{A}{4}\right)},$$

hence results (6).
We get (7) out of (6) considering that $a=2R\sin A$. □

Corollary 1. *In the ABC triangle, $\mu(\widehat{A})=\dfrac{\pi}{2}$, we have*

$$\frac{R}{2}\ge\sqrt{2}+1\quad\textit{(Emmerich's inequality)}\tag{8}$$

and equality can be obtained if and only if ABC is an isosceles triangle.

Proof. We obtain it out of (5) or (7) because $\mu(\widehat{A}) = \dfrac{\pi}{2}\,.$ □

Observation 1. Inequality (5) is proved by [2] and [4].

Proposition 3. *In any ABC triangle we have:*

$$r \geq \frac{1}{2}\left(\sqrt{2a^2-(b-c)^2}-a\right) \quad \textit{if and only if} \quad \mu(\widehat{A}) \leq \frac{\pi}{2} \tag{9}$$

$$r \leq \frac{1}{2}\left(\sqrt{2a^2-(b-c)^2}-a\right) \quad \textit{if and only if} \quad \mu(\widehat{A}) \geq \frac{\pi}{2} \tag{10}$$

$$r = p-a \quad \textit{if and only if} \quad \mu(\widehat{A}) = \frac{\pi}{2}\,. \tag{11}$$

Proof. Considering the function $f : (0,\infty) \to \mathbb{R}$,

$$f(x) = \frac{1}{2}\left(\sqrt{a^2(1+x^2)-(b-c)^2}-ax\right), \quad \forall\, x \in (0,\infty)\,.$$

We have that

$$f'(x) = \frac{1}{2}\left(\frac{a^2x}{\sqrt{a^2(1+x^2)-(b-c)^2}}-a\right), \quad \forall\, x \in (0,\infty)\,.$$

From $f'(x) = 0$,results $ax = \sqrt{a^2(1+x^2)-(b-c)^2}$,which is equivalent to $a^2-(b-c)^2 = 0$, equality that no longer takes place because $a > |b-c|$. Since $f'(x) \neq 0$, $\forall\, x \in (0,\infty)$, results that f' is a constant sign on $(0,\infty)$. We have the variation chart in figure 1. $f(1) = \dfrac{1}{2}\left(\sqrt{2a^2-(b-c)^2}-a\right)$,

x	0					1			$+\infty$
$f'(x)$		-	-	-	-	-	-	-	-
$f(x)$	$\frac{1}{2}\sqrt{a^2-(b-c)^2}$				↘	$f(1)$		↘	0

FIG 1. Variation chart

and hence the relations (9)-(11). □

Observation 2. Inequalities (9), (10) are equal if and only if $\mu(\widehat{A}) = \dfrac{\pi}{2}\,.$

Proposition 4. *In any ABC triangle we have:*

$$\frac{R}{2} \leq \frac{1}{\sqrt{2\sin^2 A - (\sin B - \sin C)^2 - \sin A}}$$
$$\textit{if and only if} \quad \mu(\widehat{A}) \leq \frac{\pi}{2} \tag{12}$$

$$\frac{R}{r} \geq \frac{1}{\sqrt{2\sin^2 A - (\sin B - \sin C)^2} - \sin A}$$
$$\textit{if and only if} \quad \mu(\widehat{A}) \geq \frac{\pi}{2} \tag{13}$$

$$\frac{R}{r} = \frac{1}{\sqrt{2 - (\sin B - \sin C)^2} - 1} \quad \textit{if and only if} \quad \mu(\widehat{A}) = \frac{\pi}{2}\,. \tag{14}$$

Proof. By changing $a = 2R\sin A$ and the analogous, in relations from Proposition 3 we get Proposition 4. □

Observation 3. Relations (12)-(14) can be found in [2].

Proposition 5. *In any ABC triangle we have*

$$\frac{R}{r} \geq \frac{1}{\sqrt{2\sin^2 A - (\sin B - \sin C)^2} - \sin A \operatorname{tg}\frac{A}{2}}$$
$$\textit{if and only if} \quad \mu(\widehat{A}) \leq \frac{\pi}{2} \quad \textit{and} \quad B = C\,. \tag{15}$$

$$\frac{R}{r} \geq \frac{1}{\sin A\left(\sqrt{2} - \operatorname{tg}\frac{A}{2}\right)} \quad \textit{if and only if} \quad \mu(\widehat{A}) \leq \frac{\pi}{2}\,. \tag{16}$$

Proof. Inequality (15) can be obtained from inequality (2.). Inequality (16) can be obtained from (15) considering that $(\sin B - \sin C)^2 \geq 0$. □

Corollary 2. *In any kind of ABC triangle takes place the relation*

$$\frac{1}{\sin A\left(\sqrt{2} - \operatorname{tg}\frac{A}{2}\right)} \leq \frac{1}{2\sin\frac{A}{2}\left(1 - \sin\frac{A}{2}\right)} \leq \frac{R}{r} \quad \textit{if} \quad \mu(\widehat{A}) \leq \frac{\pi}{2}\,. \tag{17}$$

Proof. Considering (5) it is enough to prove that

$$\frac{1}{\sin A\left(\sqrt{2} - \operatorname{tg}\frac{A}{2}\right)} \leq \frac{1}{2\sin\frac{A}{2}\left(1 - \sin\frac{A}{2}\right)} \quad \text{if} \quad \mu(\widehat{A}) \leq \frac{\pi}{2}\,.$$

We have $\sin A\left(\sqrt{2}-\operatorname{tg}\frac{A}{2}\right)\geq 2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)$, equivalent to $2\sin\frac{A}{2}\left(\sqrt{2}\cos\frac{A}{2}-\sin\frac{A}{2}\right)\geq 2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)$, equivalent to $\cos\frac{A}{2}\geq\frac{\sqrt{2}}{2}$, equivalent to $\frac{A}{2}\leq\frac{\pi}{4}$, hence $\mu(\widehat{A})\leq\frac{\pi}{2}$. □

Proposition 6. *In any ABC triangle we have*

$$r_a = r + a\ \operatorname{tg}\frac{A}{2} \tag{18}$$

$$r_a=\frac{1}{2}\left(\sqrt{a^2\left(1+\operatorname{tg}^2\frac{A}{2}\right)-(b-c)^2}+a\ \operatorname{tg}\frac{A}{2}\right) \tag{19}$$

and

$$\frac{R}{r_a}=\frac{1}{\sqrt{\sin^2 A\left(1+\operatorname{tg}^2\frac{A}{2}\right)-(\sin B-\sin C)^2}+\sin A\ \operatorname{tg}\frac{A}{2}} \tag{20}$$

$$\frac{R}{r_a}=\frac{1}{2\sin\frac{A}{2}\left(\cos\frac{B-C}{2}+\sin\frac{A}{2}\right)} \tag{21}$$

Proof. We prove relation (18) considering $r=\frac{s}{p}$ and $r_a=\frac{s}{p-a}$. We obtain relation (19)from relations (1) and (18). Identity (20) can be obtained from (19) by changing sides a, b, c in sine theorem. Identity (21) can be obtained from identity (20). □

Proposition 7. *In any kind of ABC triangle we know*

$$\frac{R}{r_a}\geq\frac{1}{2\sin\frac{A}{2}\left(1+\sin\frac{A}{2}\right)} \tag{22}$$

$$r_a=\frac{a}{2}\operatorname{tg}\left(\frac{\pi}{4}+\frac{A}{4}\right) \tag{23}$$

$$\frac{R}{r_a}\geq\frac{1}{\sin A\ \operatorname{tg}\left(\frac{\pi}{4}+\frac{A}{4}\right)} \tag{24}$$

and equality in any of (22) – (24) *inequalities can be obtained if and only if* $B=C$.

Proof. Considering $0 \leq \cos\dfrac{B-C}{2} \leq 1$, from relation (21) results (22). From (19) results that $r_a \leq \dfrac{1}{2}\left(\sqrt{a^2\left(1+\operatorname{tg}^2\dfrac{A}{2}\right)}+a\operatorname{tg}\dfrac{A}{2}\right)$ is equivalent to $r_a \leq \dfrac{a\left(1+\sin\dfrac{A}{2}\right)}{2\cos\dfrac{A}{2}}$. Similar to proving inequalities (6) and (7), we obtain inequalities (23) and (24). □

Corollary 3. *In the ABC triangle, $\mu(\widehat{A}) = \dfrac{\pi}{2}$ we have*

$$\frac{R}{r_a} \geq \sqrt{2}-1 \tag{25}$$

and equality can be obtained if and only if ABC is a right isosceles triangle.

Proof. It can be obtained from inequalities (22) or (24) for $\mu(\widehat{A}) = \dfrac{\pi}{2}$. □

Corollary 4. *In the ABC triangle, $\mu(\widehat{A}) = \dfrac{\pi}{2}$, takes place*

$$R^2 \geq r \cdot r_a \tag{26}$$

and equality is obtained if and only if ABC is a right isosceles triangle.

Proof. It results from inequalities (8) and (25). □

Proposition 8. *In any ABC triangle we have*

$$r_a \leq \frac{1}{2}\left(\sqrt{2a^2-(b-c)^2}+a\right) \quad \textit{if and only if} \quad \mu(\widehat{A}) \leq \frac{\pi}{2} \tag{27}$$

$$r_a \geq \frac{1}{2}\left(\sqrt{2a^2-(b-c)^2}+a\right) \quad \textit{if and only if} \quad \mu(\widehat{A}) \geq \frac{\pi}{2} \tag{28}$$

$$r_a = p \quad \textit{if and only if} \quad \mu(\widehat{A}) = \frac{\pi}{2}. \tag{29}$$

Proof. Considering the function $f : (0,\infty) \to \mathbb{R}$,

$$f(x) = \frac{1}{2}\left(\sqrt{a^2(1+x^2)-(b-c)^2}+ax\right), \quad \forall\, x \in (0,\infty).$$

We know that

$$f'(x) = \frac{a^2x}{\sqrt{a^2(1+x^2)-(b-c)^2}}+a > 0, \quad \forall\, x \in (0,\infty)$$

and we have the variation chart in figure 2.

x	0		1		+∞
$f'(x)$	+ +	+	+	+ +	+
$f(x)$	$\sqrt{a^2-(b-c)^2}$	↗	$f(1)$	↗	$+\infty$

FIG 2. Variation chart

$$f(1) = \frac{1}{2}\left(\sqrt{2a^2-(b-c)^2}+a\right),$$

Hence we get relations (27)-(29). □

Proposition 9. *In any ABC triangle we have*

$$\frac{R}{r_a} \geq \frac{1}{\sqrt{2\sin^2 A-(\sin B-\sin C)^2}+\sin A} \quad \textit{if and only if} \quad \mu(\widehat{A}) \leq \frac{\pi}{2} \tag{30}$$

$$\frac{R}{r_a} \leq \frac{1}{\sqrt{2\sin^2 A-(\sin B-\sin C)^2}+\sin A} \quad \textit{if and only if} \quad \mu(\widehat{A}) \geq \frac{\pi}{2} \tag{31}$$

$$\frac{R}{r_a} = \frac{1}{\sqrt{2-(\sin B-\sin C)^2}+\sin A} \quad \textit{if and only if} \quad \mu(\widehat{A}) = \frac{\pi}{2}. \tag{32}$$

Proof. By changing $a = 2R\sin A$ and analogous in relation from Proposition 8, we obtain Proposition 9. □

Corollary 5. *In any ABC triangle we have*

$$\sqrt{2}-1 \leq \frac{\sqrt{2}-1}{\sin A} \leq \frac{1}{\sqrt{2\sin^2 A-(\sin b-\sin c)^2}+\sin A} \leq \tag{33}$$

$$\leq \frac{R}{r_a} \;\textit{if}\; \mu\left(\widehat{A}\right) \leq \frac{\pi}{2}$$

Proof. Inequality $\sqrt{2}-1 \leq \frac{\sqrt{2}-1}{\sin A}$ is true.

Since $(\sin B - \sin C)^2 \geq 0$ results $\frac{1}{\sqrt{2\sin^2 A - (\sin b - \sin c)^2} + \sin A} \geq$

$\geq \frac{1}{\sqrt{2\sin^2 A} + \sin A} = \frac{1}{\sin A\left(\sqrt{2}+1\right)} = \frac{\sqrt{2}-1}{\sin A}$ and we also take into account (30). □

Observation 4. We obtain Corollary 3 from Corollary 5.

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