**CREATIVE MATH.**
**12** (2003), 59 - 64

# A matrix method for obtaining the spline collocation function

Magnolia Rebeleş

Abstract. In this paper it is presented a procedure for obtaining a spline collocation function which approximate the solution of the initial value problem in linear differential equation, using a matrix method. There are also presented some results of implementing a *Borland C* program for a linear differential equation with initial values.

## 1. Introduction

In the last years, the problem of approximating the solution of linear and non linear differential equations by spline functions has been of growing interest. Many authors [1],[2],[3],[4] have proposed various methods to approximate the solution by means of spline. In this paper, it is constructed a matrix algorithm for obtaining the coefficients of the spline collocation function starting from its definition. Based on this algorithm, I construct a *Borland C* program to compute these coefficients and to compute the values of those spline collocation functions. But this result is restricted to a linear differential equation.

Consider the $n$-th order differential equation

$$y^{(n)} = f(x, y, y^{(1)}, ..., y^{(n-1)}) \tag{1.1}$$

with initial conditions:

$$y(x_0) = y_0, y'(x_0) = y'_0, ..., y^{(n-1)}(x_0) = y_0^{(n-1)}, x_0 \in [a, b] \tag{1.2}$$

where $f : [a, b] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous with respect to $y(x), y'(x), ..., y^{(n-1)}(x), x \in [a, b]$ and $f$ satisfies a Lipschitz condition in the last $n$ arguments. The continuity of $f$ and the Lipschitz condition guarantee the existence and uniqueness of the solution $y : [a, b] \longrightarrow \mathbb{R}$ of the problem (1.1),(1.2).

The spline $s$ which approximates the exact solution $y$ is of degree $m \geq n+1$ and class $C^{m-1}[a, b]$.

Let $x_k, k = 0, ..., N$ be equal spaced points in $[a, b]$, with

$$x_0 = a, x_{N+1} = b,$$
$$x_k - x_{k-1} = h, k = 1, ..., N + 1.$$

Received: 25.05.2003; In revised form: 15.11.2003
Key words and phrases. *Spline collocation function.*

Denote $I_{k+1} := [x_k, x_{k+1}], k = 0, ..., N$. On the interval $I_{k+1}$ the spline collocation function is defined as follows

$$s_{k+1}(x) = \sum_{j=0}^{m-1} \frac{s_k^j(x_k)}{j!}(x-x_k)^j + \frac{a_{k+1}}{m!}(x-x_k)^m, k = 0, ..., N, \tag{1.3}$$

where

$$s_0^{(j)}(x_0) = y^{(j)}(x_0), j = 0, ..., m-1$$

The relations (1.3) give equations with unknown values $a_{k+1}, k = 0, ..., N$. These real parameters are obtained from the collocation conditions

$$s_{k+1}^{(n)}(x_{k+1}) = f(x_{k+1}, s_{k+1}(x_{k+1}), s_{k+1}^{(1)}(x_{k+1}), ..., s_{k+1}^{(n-1)}(x_{k+1})), \tag{1.4}$$

$k = 0, ..., N$

In [3] it is presented the existence and uniqueness of the spline function.

## 2. Main result

2.1. **The construction of the coefficients of the collocation spline function.** Using the equality

$$\{\frac{s_k^{(j)}(x_k)}{j!} \cdot (x-x_k)^j\}^{(r)} = \frac{s_k^{(j)}(x_k)}{(j-r)!} \cdot (x-x_k)^{j-r},$$

the matrix of the coefficients of the function $s_1(x), x \in I_1$ and its derivatives $s_1, s_1^{(1)}, ..., s_1^{(n)}$ is $S_1$

| $S_1$ | | | | | | |
|---|---|---|---|---|---|---|
| $s_1^{(0)}$ | $y(x_0)$ | $\frac{y'(x_0)}{1!}$ | $\frac{y''(x_0)}{2!}$ | ... | $\frac{y^{(m-1)}(x_0)}{(m-1)!}$ | $\frac{a_1}{m!}$ |
| $s_1^{(1)}$ | 0 | $y'(x_0)$ | $\frac{y''(x_0)}{1!}$ | ... | $\frac{y^{(m-1)}(x_0)}{(m-2)!}$ | $\frac{a_1}{(m-1)!}$ |
| $s_1^{(2)}$ | 0 | 0 | $y''(x_0)$ | ... | $\frac{y^{(m-1)}(x_0)}{(m-3)!}$ | $\frac{a_1}{(m-2)!}$ |
| ... | ... | | | | | |
| $s_1^{(n)}$ | 0 | 0 | 0 | ... | $\frac{y^{(m-1)}(x_0)}{(m-n-1)!}$ | $\frac{a_1}{(m-n)!}$ |

| $H$ | | | | |
|---|---|---|---|---|
| | 1 | 0 | ... | 0 |
| | $h$ | 1 | ... | 0 |
| | $h^2$ | $h$ | ... | 0 |
| | ... | | | |
| | $h^m$ | $h^{m-1}$ | ... | $h^{m-n}$ |

where the initial values $y(x_0), ..., y^{(n-1)}(x_0)$ are known from (1.2) and the next $y^{(n)}(x_0), ..., y^{(m-1)}(x_0)$ have to be computed. Every fraction has the denominator an increasing factorial $j!$ if we complete the matrix lines, and has a decreasing factorial if we complete the matrix columns, taking into account that if we obtain $j < 0$ we complete automatically with zero. The last column contains the unknown value $a_1$. The matrix $S_1$ has $(m+1) \times (n+1)$ degree.

The elements of $H$ matrix are the values of the function

$$(x - x_k)^j$$

where $x := x_{k+1}, k = 0, ..., N, j = 0, ..., m-1$ and its derivatives up to the $n$-th order; the last line has the values

$$(x_{k+1} - x_k)^{m-i} := h, k = 0, ..., N, i = 0, ..., n.$$

Therefore, the matrix $H$ has $(n+1) \times (m+1)$ degree.

The unknow value $a_1$ from $S_1$ matrix is given by substituting the product of each line from $S_1$ with the corresponding column from $H$ in (1.3).

In a similar manner, we can construct the matrix $S_{k+1}, k = 1, ..., N$ on each subinterval $I_{k+1}$

$$\begin{array}{c|ccccccc}
S_{k+1} & & & & & & \\
\hline
s_{k+1}^{(0)} & s_k(x_k) & \frac{s'_k(x_k)}{1!} & \frac{s''(x_k)}{2!} & ... & \frac{s_k^{(m-1)}(x_k)}{(m-1)!} & \frac{a_{k+1}}{m!} \\
s_{k+1}^{(1)} & 0 & s'_k(x_k) & \frac{s''(x_k)}{1!} & ... & \frac{s_k^{(m-1)}(x_k)}{(m-2)!} & \frac{a_{k+1}}{m-1!} \\
s_{k+1}^{(2)} & 0 & 0 & s''(x_k) & ... & \frac{s_k^{(m-1)}(x_k)}{(m-3)!} & \frac{a_{k+1}}{m-2!} \\
... & ... & & & & & \\
s_{k+1}^{(n)} & 0 & 0 & 0 & ... & \frac{s_k^{(m-1)}(x_k)}{(m-n-1)!} & \frac{a_{k+1}}{(m-n)!}
\end{array}$$

The product of each line from $S_{k+1}$ with the corresponding column from $H$ gives us the values $s_k(x_k), s_k^{(1)}(x_k), ..., s_k^{(n)}(x_k)$ necessary to construct the next spline $s_{k+1}(x)$.

2.2. **Practical results using the computer.** Three problems are proposed in this section:

1. The implementation of a routine which calculates the coefficients of the spline collocation function
2. The representation of the exact solution and its spline approximation (using MATLAB)
3. To apply the result to the initial value problem

$$y'' = \frac{1}{2}y' + \frac{1}{2}y, x \in [0, 1]$$

$$y(0) = 1, y'(0) = 1.$$

using a cubic spline function ($m = 3$) for the approximation of the exact solution $y(x) = e^x, x \in [0, 1]$, when the step size is $h = 0.25$.

The algorithm calculates the coefficients of a $m$-degree spline function and the values of the spline collocation function.

`Input:` $n =$ the order of the linear differential equation with constant coefficients

$m =$ the degree of the collocation spline function
$[a, b] =$ the interval of the variational problem
$h =$ the step of the devision
$x[k] =$ the knots of the devision, $k = 0, N$
$N =$ the number of the knots
$c[i] =$ the coefficients of $i$ order derivatives of $y(x)$ function
$y_0[i] =$ the initial value of $i$ order derivatives of $y(x)$ function

`Output:` $s[k] =$ the coefficients of the matrix which has $i$ order derivatives on $i$ line $i = 0, ..., n$ of the spline collocation function defined on $I_k$

$s(x) =$ the vector of the values of the collocation function calculated in $x \in [x_{k-1}, x_k]$

`Step 1:` The introduction of the initial value problem

$$y^{(n)} = c_{n-1}y^{n-1} + c_{n-2}y^{(n-2)} + ... + c_1y' + c_0y \tag{2.5}$$

$$y^{(i)}(x_0) = y_0^{(i)}, i = 0, ..., n-1$$

`Step 2:` The calculate of the knots of the uniform partition $\Delta$ :

$$\Delta : [a = x_0 < x_1 < ... < x_N < x_{n+1} = b]$$

`Step 3:` The calculate of the values of the derivatives up to the order $m \geq n$ in $x_0$: for $i = 0, ..., n-1$, $y_0^{(i)}$ are known

for $i = n$, $y^{(n)}(x_0)$ are calculated from (1.1);
for $i = n+1, ..., m-1$ are obtained by derivate (1.1).

`Step 4:` The construction of the matrix $s[k]$. The elements of the matrix are calculated using the matrix method described in 2.1.

`Step 4.1:` The construction of $s[1]$ matrix
`Step 4.2:` The construction of $X$ matrix
`Step 4.3:` The calculate of $A[1]$ constant
`Step 4.4:` The list of $s[1]$ matrix
`Step 4.5:` The calculate of the elements of $s[k]$ matrix
`Step 4.6:` The matrix $s[k]$

`Step 5:` The compute of $s(x)$ values.

For the vector of the equal spaced points in $[0, 1]$, with $H = 0.1$ step size, together with the points $x_k, k = 0, ..., 4, h = 0.25$

$$OX = [0\ \ 0.1\ \ 0.2\ \ 0.25\ \ 0.3\ \ 0.4\ \ 0.5\ \ 0.6\ \ 0.7\ \ 0.75\ \ 0.8\ \ 0.9\ \ 1]$$

and for $OY$ vector of $s, s', s''$ spline collocation functions (see tables 1,2,3) we have the graphic representations of the approximate solutions, $s, s', s''$, together with the exact solution, $y$. The pairs $(OX[i], OY[i])$ are marked with $'+'$.

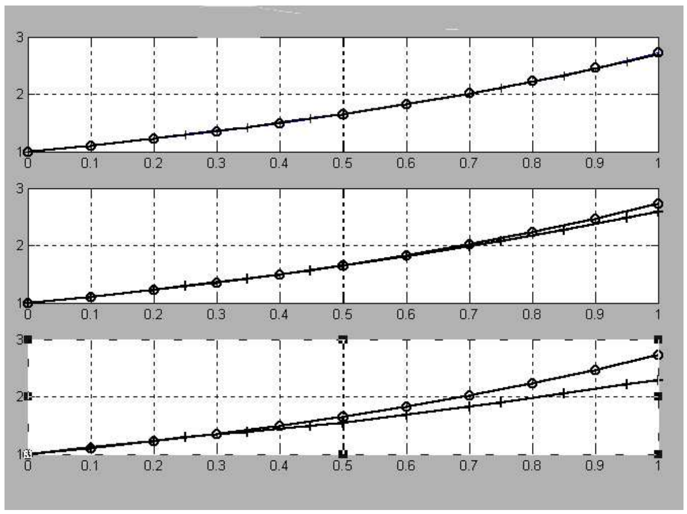

FIGURE 1. s,s',s" functions and the exact solution y (m=3)

| $s$ | | | |
|---|---|---|---|
| | $oY[1][0] = 1.0000000$ | $oY[1][1] = 1.1051899$ | $oY[1][2] = 1.2215196$ |
| | $oY[2][3] = 1.2842178$ | $oY[2][4] = 1.4193705$ | $oY[2][5] = 1.5683724$ |
| | $oY[3][6] = 1.6483792$ | $oY[3][7] = 1.8201073$ | $oY[3][8] = 2.0086491$ |
| | $oY[4][9] = 2.1096833$ | $oY[4][10] = 2.3262067$ | $oY[4][11] = 2.5632904$ |
| | $oY[4][12] = 2.6900263$ | | |

TABLE 1. The values of s for m=3

| $s'$ | | | |
|---|---|---|---|
| | $oY[1][0] = 1.0000000$ | $oY[1][1] = 1.1056983$ | $oY[1][2] = 1.2227933$ |
| | $oY[2][3] = 1.2856145$ | $oY[2][4] = 1.4191061$ | $oY[2][5] = 1.5625976$ |
| | $oY[3][6] = 1.6380935$ | $oY[3][7] = 1.7989088$ | $oY[3][8] = 1.9743718$ |
| | $oY[4][9] = 2.0675960$ | $oY[4][10] = 2.2654531$ | $oY[4][11] = 2.4788032$ |
| | $oY[4][12] = 2.6900263$ | | |

TABLE 2. The values of s' for m=3

| $s''$ | | | |
|---|---|---|---|
| | $oY[1][0] = 1.0000000$ | $oY[1][1] = 1.1139665$ | $oY[1][2] = 1.2279329$ |
| | $oY[2][3] = 1.2849162$ | $oY[2][4] = 1.3849162$ | $oY[2][5] = 1.4849162$ |
| | $oY[3][6] = 1.5349162$ | $oY[3][7] = 1.6813910$ | $oY[3][8] = 1.8278658$ |
| | $oY[4][9] = 1.9011033$ | $oY[4][10] = 2.0560360$ | $oY[4][11] = 2.2109687$ |
| | $oY[4][12] = 2.2884350$ | | |

TABLE 3. The values of s" or m=3

## REFERENCES

[1] Callender E.D.,*Single step methods and low order splines for solutions of ordinary differential equations*, SIAM J. Numer. Anal. **8** (1971), 61-66

[2] Fawzy T. and Soliman S., *A spline approximation method for the initial value problem* $y^{(n)} = f(x, y, y')$, Annales Univ.Sci., Budapst, Sect. Comp., **10** (1990), 299-323

[3] Micula G., *Spline Functions and Applications* (Romanian), Editura Tehnică, Bucureşti, 1978

[4] Sovegjarto A., *Spline approximations to solutions of initial value problems*, Annales. Univ. Sci. Budapest., Sect.Comp., **16** (1996), 359-376

COLEGIUL ECONOMIC "NICOLAE TITULESCU"
PROGRESULUI 45, 430291 BAIA MARE
ROMANIA
*E-mail address*: m_rebeles@yahoo.com