CREATIVE MATH.
12 (2003), 00-00

The Window Fourier transform – a suitable alternative for image compression?

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Abstract. The *Discrete Cosine Transform* [6], [8] on which the best-known image compression schemes are based [7], [1], has the major disadvantage that it does not localize the frequency components in time. The *Window Fourier Transform* (WFT) [3], [4], [5] solves this problem, but has limitations related to the resolutions in time and frequency. This article contains a visual evaluation of the WFT characteristics.

1. The Window Fourier Transform

In order to solve the time localization of the frequency components, the analyzed signal is divided through the *Window Fourier Transform* (WFT) in intervals, small enough for the signal to be stationary within each of the intervals, and then the intervals are processed separately. For this purpose, a window function w is used, that is non-zero over a finite interval, and has finite integral. The length of the intervals corresponds to the width of the window.

The window function can be for example the Gaussian function defined in (1). The width of the window is determined by the parameter a.

$$w(t) = e^{-a\frac{t^2}{2}} \tag{1}$$

The computing of the WFT for a signal $x(t)$ involves more steps in which the signal and the window function are multiplied, and then the *Fourier Transform* (FT) of the product is computed and the window function is shifted in time. While computing the WFT, the window function crosses the signal from one end to the other. The result is a surface that reveals information about the spectral components of the signal and their time localization.

The WFT of function $x(t)$ is defined in (2), and the inverse transform is given in (3), where c is a constant that depends on the window function $w(t)$, and * denotes the complex conjugate.

Received: 17.05.2003; In revised form:
Key words and phrases. Window Fourier Transform.

$$X_{WFT}^{w}(\tau, f) = \int_{\mathbb{R}} x(t) w^{*}(t-\tau) e^{-2j\pi f t} dt \tag{2}$$

$$x(t) = c \iint_{\mathbb{R}\times\mathbb{R}} X_{WFT}^{w}(\tau, f) w(t-\tau) e^{2j\pi f t} d\tau df \tag{3}$$

The WFT is in fact the FT of the signal $x(t)$, computed after applying the filter $w(t-\tau)$. The result is a two-parameter function, in which the second parameter τ represents the position of the filter.

2. Limitations

The function $w(t)$ allows the detection of a window in the spectrum of $x(t)$, localized around τ. Thus the problem of localizing the signal components in time was solved, but there is a limitation: The time and frequency resolutions cannot be enhanced simultaneously however much.

The bandwidth Δf and the width Δt of the window function $w(t)$ are given in relations (4) and (5), where $W(f)$ is the FT of $w(t)$ [2].

$$\Delta f^{2} = \frac{\int_{\mathbb{R}} f^{2} \left| W(f) \right|^{2} df}{\int_{\mathbb{R}} \left| W(f) \right|^{2} df} \tag{4}$$

$$\Delta t^{2} = \frac{\int_{\mathbb{R}} t^{2} \left| w(t) \right|^{2} dt}{\int_{\mathbb{R}} \left| w(t) \right|^{2} dt} \tag{5}$$

The WFT is not able to discriminate two δ impulses in time, if they are closer than Δt. If the signal $x(t)$ contains the sum of two pure sinusoids, that are equivalent to two δ impulses in frequency, the WFT is not able to discriminate them, if they are closer than Δf. Within an analysis, the time and frequency resolutions can be independently enhanced. For good frequency resolution, a wide window function has to be used, which has a small bandwidth, and for good time resolution a narrow window has to be used, which has a large bandwidth. The two characteristics, Δf and Δt are bound by the *Heisenberg inequality* that is given in (6).

$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi} \tag{6}$$

Once the window function is established, the time and frequency resolutions remain fixed for all the regions of the WFT surface. There is no possibility to adapt the analysis. If for example a good time resolution is wanted for the sharp discontinuities of the signal, the frequency resolution of the long smooth components has to be sacrificed.

3. An example

For a visual evaluation of the WFT characteristics, figures 1,2,3 and 4 present the graph of the WFT computed for the function $x_1(t)$ defined in (7), with the window function defined in (1), for several values of parameter a, that controls it's bandwidth. Each of the graphs is presented in 3D and in top views.

$$x_1(t) = \begin{cases} \cos(2\pi t), & t \in [0, 4\pi] \\ \cos(4\pi t), & t \in (4\pi, 8\pi] \\ \cos(8\pi t), & t \in (8\pi, 12\pi] \\ 0, & t \notin [0, 12\pi] \end{cases} \tag{7}$$

The correspondence between the bandwidth of the window function and the time and frequency resolutions of the WFT can be easily observed. The first graph was generated with a relatively wide window function (a = 0.01), which corresponds to a small bandwidth. In consequence the WFT has good frequency resolution and poor time resolution. The last of the graphs is situated at the opposite pole. In this case a relatively narrow window function, that has a large bandwidth, was used for generating the WFT surface ($a = 10$). In this case the WFT has good time resolution and poor frequency resolution.

For solving the resolution problem, the bandwidth of the window function could be modified with the frequency. The purpose of this artifice is the increase of time resolution for the sharp discontinuities, while keeping a good frequency resolution for the long and smooth components. If the signal contains long high frequency components, such as different kinds of noise, this strategy will not be efficient. But in the case of natural images, which are usually composed by large smooth areas separated by localized discontinuities, this technique could be very efficient. The *Wavelet Transform* [8] uses this strategy.

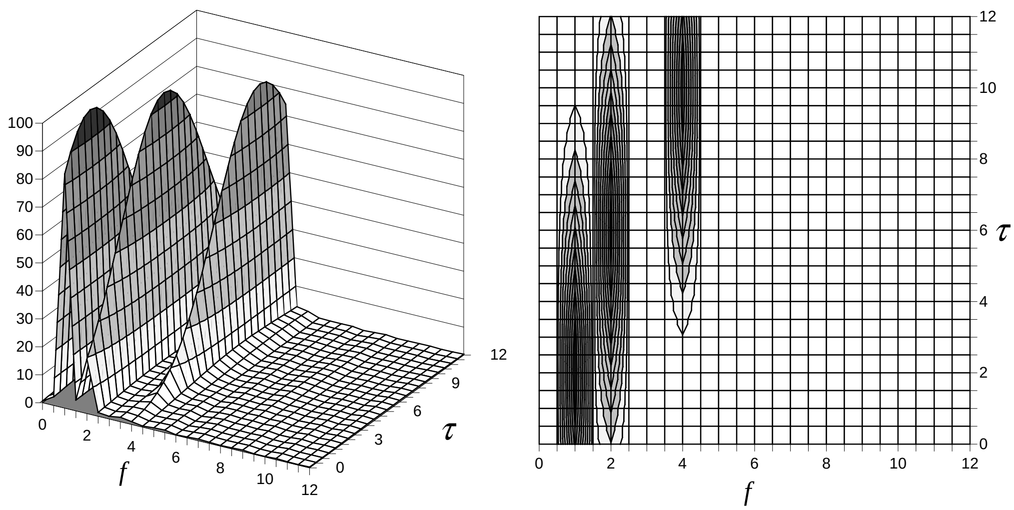

Figure 1: The WFT surface for $a = 0.01$. The frequency resolution is good, but the time resolution is poor.

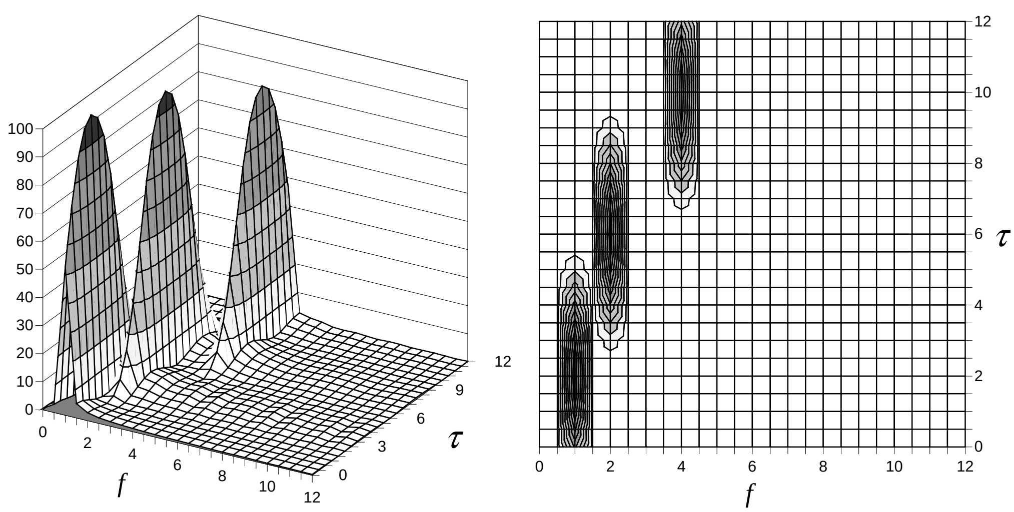

Figure 2: The WFT surface for $a = 0.1$. The time resolution is better than in the previous case.

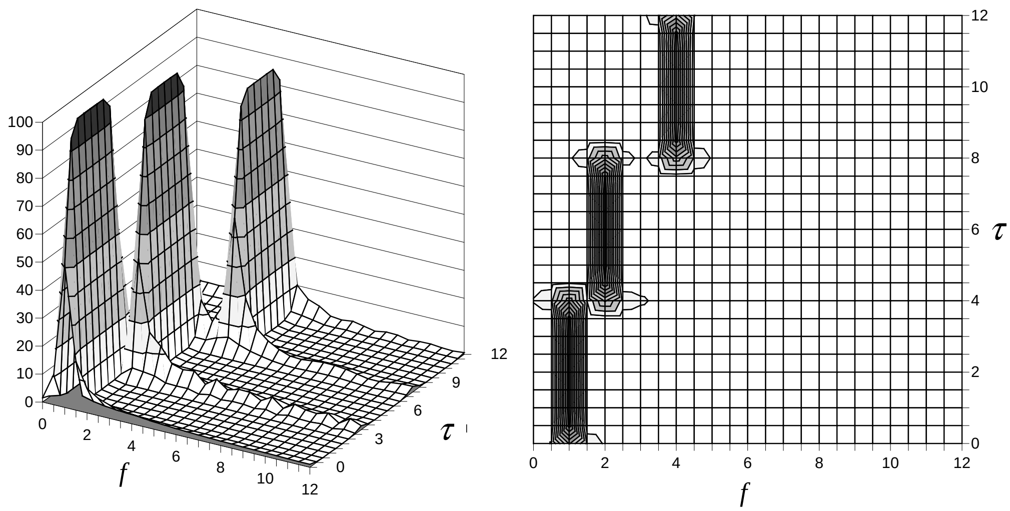

Figure 3: The WFT surface for $a = 1$. The time resolution is better than in the previous case, but the frequency resolution worsend.

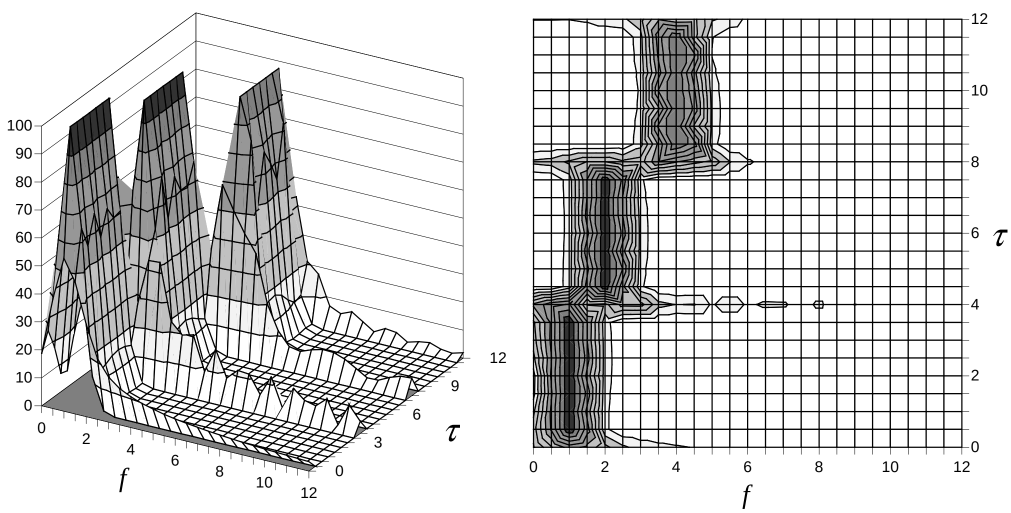

Figure 4: The WFT surface for $a = 10$. The time resolution is very good, but the frequency resolution is weak.

References

[1] Cosma O., *Contributions to the Coding of Image Subbands*, thesis, Politehnica University Bucharest, 2003.
[2] Fournier A., *Wavelets and their Applications in Computer Graphics*, SIGGRAPH Course Notes, University of British Columbia 1995.
[3] Polikar R., *The Fourier Transform and the Short Term Fourier Transform*, http://users.rowan.edu/∼polikar/WAVELETs/WTpart2.html.
[4] Schniter Ph., *Short-time Fourier Transform*, http://cnx.rice.edu/ content/ m10417/ latest/.
[5] Selesnick I., *Short Time Fourier Transform*, http://cnx.rice.edu/ content/m10570/ latest/.
[6] Signal Models Algebra Representations & Transforms, *Discrete Cosine Transform, Type II and III*, www.ece.cmu.edu/∼smart/examples/algogen/dct2.html.
[7] Wallace G. K., *The JPEG Still Picture Compression Standard*, Communications of the ACM, **34**, Issue 4, 04.1991.
[8] Wikipedia, *Discrete Cosine Transform*, http://www.wikipedia.org/wiki/Discrete_ cosine_transform.

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