CREATIVE MATH.
13 (2004), 1 - 4

On the rate of convergence of Archimedean double sequences

Iulia Costin and Gheorghe Toader

Abstract. Let M and N be two means. The pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by

$$a_{n+1} = M(a_n, b_n), b_{n+1} = N(a_{n+1}, b_n), n \geq 0,$$

is called an Archimedean double sequence. We study the rate of convergence of these sequences to a common limit.

1. Introduction

There are more definitions of means. We use here the following one.

Definition 1. A **mean** (on the interval J) is defined as a function $M : J^2 \to J$, which has the property

$$\min(a, b) \leq M(a, b) \leq \max(a, b),\ \forall a, b \in J\ .$$

The mean M is called **symmetric** if

$$M(a, b) = M(b, a),\ \forall a, b \in J\ .$$

We shall refer to weighted Gini means, defined by

$$\mathcal{B}_{r,s;\lambda}(a, b) = \left[\frac{\lambda \cdot a^r + (1-\lambda) \cdot b^r}{\lambda \cdot a^s + (1-\lambda) \cdot b^s}\right]^{\frac{1}{r-s}}\ ,\ r \neq s$$

and

$$\mathcal{B}_{r,r;\lambda}(a, b) = \left[a^{\lambda \cdot a^r} . b^{(1-\lambda)\cdot b^r}\ \right]^{\frac{1}{\lambda \cdot a^r + (1-\lambda)\cdot b^r}}$$

with $\lambda \in [0, 1]$ fixed. They are symmetric only for $\lambda = 1/2$. If $s = 0$ we get the special case of weighted power means.

We shall use the following results regarding the partial derivatives of means and which can be found in [7].

Theorem 1. *If M is a differentiable mean then*

$$M_a(c, c) + M_b(c, c) = 1 \tag{1}$$

and

$$0 \leq M_a(c, c) \leq 1.$$

As a special case we get the next result proved in [4].

Received: 19.09.2004. In revised form: 21.12.2004.
2000 *Mathematics Subject Classification.* 26E60.
Key words and phrases. *Archimedean double sequences, rate of convergence.*

Corollary 1. *If M is a symmetric differentiable mean then*

$$M_a(c,c) = M_b(c,c) = 1/2.$$

Remark 1. For non symmetric means, this property is not valid. For example, for $M = \mathcal{B}_{r,s;\lambda}$, we have

$$M_a(c,c) = \lambda.$$

2. Archimedean double sequences

The well known Archimedes' polygonal method of evaluation of π, was interpreted in [5] as a double sequence. This led to the next definition. Let us consider two means M and N defined on the interval J and two initial values $a, b \in J$.

Definition 2. The pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by

$$a_{n+1} = M(a_n, b_n) \text{ and } b_{n+1} = N(a_{n+1}, b_n) \ , \ n \geq 0 \tag{2}$$

where $a_0 = a, b_0 = b$, is called an **Archimedean double sequence**.

Definition 3. The mean M is **composible in the sense of Archimedes** (or **A-composible**) with the mean N if the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by (2) are convergent to a common limit $M \boxtimes N(a,b)$ for each $a, b \in J$.

Remark 2. In this case $M \boxtimes N$ is also a mean on J called **Archimedean compound mean** (or **A-compound mean).**

Remark 3. The classical case of Archimedes corresponds to the composition $\mathcal{H}\boxtimes\mathcal{G}$, where $\mathcal{H}$ and $\mathcal{G}$ denote the harmonic mean, respectively the geometric mean, defined by

$$\mathcal{H}(a,b) = \frac{2ab}{a+b} \ , \ \mathcal{G}(a,b) = \sqrt{ab} \ , \ \forall a, b > 0.$$

As was determined in [5], if $0 < b_0 < a_0$, the common limit of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ is

$$\mathcal{H} \boxtimes \mathcal{G}(a_0, b_0) = \frac{a_0 b_0}{\sqrt{a_0^2 - b_0^2}} \arccos \frac{b_0}{a_0} \ .$$

In Archimedes' case, as

$$a_0 = 3\sqrt{3} \text{ and } b_0 = 3\sqrt{3}/2 \ ,$$

the common limit is π .

3. Rate of convergence

In the case of classical Archimedean algorithm it is shown that the error of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ tend to zero asymptotically like $1/4^n$. In [4] is proved that this result is valid in the case of A-composition of arbitrary differentiable symmetric means. For the general case, we have the following evaluation.

For two means M and N given on the interval J and two initial values $a, b \in J$ we denote

$$\alpha = M \boxtimes N(a,b).$$

Theorem 2. *If the means M and N have continuous partial derivatives up to second order, then the errors of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ tend to zero asymptotically like*

$$[M_a(\alpha,\alpha)\cdot(1-N_a(\alpha,\alpha))]^n \,.$$

Proof. If we write

$$a_n = \alpha + \delta_n \ , \ b_n = \alpha + \varepsilon_n \ ,$$

we deduce that, as $n \to \infty$,

$$\alpha + \delta_{n+1} = M(\alpha + \delta_n, \alpha + \varepsilon_n)$$
$$= M(\alpha,\alpha) + M_a(\alpha,\alpha)\delta_n + M_b(\alpha,\alpha)\varepsilon_n + O(\delta_n^2 + \varepsilon_n^2) \ .$$

From (1) we get

$$\delta_{n+1} = M_a(\alpha,\alpha)\delta_n + [1 - M_a(\alpha,\alpha)]\,\varepsilon_n + O(\delta_n^2 + \varepsilon_n^2) \tag{3}$$

Then

$$\alpha + \varepsilon_{n+1} = N(\alpha + \delta_{n+1}, \alpha + \varepsilon_n)$$
$$= N(\alpha,\alpha) + N_a(\alpha,\alpha)\delta_{n+1} + N_b(\alpha,\alpha)\varepsilon_n + O(\delta_{n+1}^2 + \varepsilon_n^2) \ .$$

Using again (1) and (3) we have

$$\varepsilon_{n+1} = N_a(\alpha,\alpha)\,[M_a(\alpha,\alpha)\delta_n + (1 - M_a(\alpha,\alpha))\,\varepsilon_n]$$

$+[1 - N_a(\alpha,\alpha)]\,\varepsilon_n + O(\delta_n^2 + \varepsilon_n^2)$,thus

$$\varepsilon_{n+1} = M_a(\alpha,\alpha)N_a(\alpha,\alpha)\delta_n + [1 - M_a(\alpha,\alpha)N_a(\alpha,\alpha)]\,\varepsilon_n + O(\delta_n^2 + \varepsilon_n^2) \tag{4}$$

Subtracting (4) from (3) we get

$$\delta_{n+1} - \varepsilon_{n+1} = M_a(\alpha,\alpha)\,[1 - N_a(\alpha,\alpha)]\,(\delta_n - \varepsilon_n) + O(\delta_n^2 + \varepsilon_n^2) \ .$$

On the other hand, from the monotonicity of $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ we can suppose that $\delta_n > 0$ and $\varepsilon_n < 0$ for all $n > 0$. The case when $\delta_n < 0$ and $\varepsilon_n > 0$ can be treated similarly. We have

$$\frac{\varepsilon_n - \varepsilon_{n+1}}{\delta_n - \delta_{n+1}} = \frac{M_a(\alpha,\alpha)N_a(\alpha,\alpha)\,(\varepsilon_n - \delta_n) + O(\delta_n^2 + \varepsilon_n^2)}{[1 - M_a(\alpha,\alpha)]\,(\delta_n - \varepsilon_n) + O(\delta_n^2 + \varepsilon_n^2)},$$

thus

$$\varepsilon_n - \varepsilon_{n+1} = \frac{M_a(\alpha,\alpha)N_a(\alpha,\alpha)}{M_a(\alpha,\alpha) - 1}\,(\delta_n - \delta_{n+1}) + (\delta_n - \delta_{n+1})O(|\delta_n| + |\varepsilon_n|) \ .$$

Replacing n by $n+1, n+2, ..., n+p-1$ $(p \in \mathbb{N})$, adding and using the fact that δ_n and ε_n tend monotonically to zero , we obtain

$$\varepsilon_n - \varepsilon_{n+p} = \frac{M_a(\alpha,\alpha)N_a(\alpha,\alpha)}{M_a(\alpha,\alpha) - 1}\,(\delta_n - \delta_{n+p}) + (\delta_n - \delta_{n+p})O(|\delta_n| + |\varepsilon_n|) \ .$$

Letting $p \to \infty$ we get

$$\varepsilon_n = \frac{M_a(\alpha,\alpha)N_a(\alpha,\alpha)}{M_a(\alpha,\alpha) - 1}\delta_n + O(\delta_n^2 + \varepsilon_n^2) \ .$$

Using (3) we deduce that

$$\delta_{n+1} = M_a(\alpha,\alpha)\,[1 - N_a(\alpha,\alpha)]\,\delta_n + O(\delta_n^2)$$

and from (4) we have

$$\varepsilon_{n+1} = M_a(\alpha, \alpha)\,[1 - N_a(\alpha, \alpha)]\,\varepsilon_n + O(\varepsilon_n^2)\ .$$

□

Remark 4. In the case of symmetric means, we saw that

$$M_a(\alpha, \alpha) = N_a(\alpha, \alpha) = \frac{1}{2}\ ,\ \forall \alpha \in J$$

and we get the result proved in [4] :

Corollary 2. *If the means M and N are symmetric and have continuous partial derivatives up to second order, then the error of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ tend to zero asymptotically like $1/4^n$.*

Example 1. For $M = \mathcal{B}_{r,s;\lambda}$ and $N = \mathcal{B}_{p,q;\mu}$, the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ tend to zero asymptotically like $[\lambda\,(1-\mu)]^n$.

Remark 5. In [2] will be given a method of acceleration of the convergence, that is, it will be constructed a combination of the sequences which converges faster than each of them. For some symmetric means, such a method was used even in [6].

References

[1] Bullen, P. S., *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht/ Boston/ London, 2003

[2] Costin, I., *Acceleration of the convergence of Archimedean double sequences*, Mathematical Analysis and Approximation Theory (2004), Mediamira Science Publisher (to appear)

[3] Foster, D. M. E., Phillips, G. M., *The approximation of certain functions by compound means*, Proc. of NATO Adv. Study Inst., St. John's Newfoundland, 1983

[4] Foster, D. M. E., Phillips, G. M., *A generalization of the Archimedean double sequence*, J. Math. Anal. Appl., **110**(2) (1984), 575-581

[5] Phillips, G. M., *Archimedes the numerical analyst*, Amer. Math. Monthly **88**(3) (1981), 165-169

[6] Snell, W., *Cyclometricus*, Leyden, 1621

[7] Toader, S., *Derivatives of generalized means*, Math. Inequ. & Appl. **5**(3) (2002), 517-523

Technical University of Cluj-Napoca
Department of Computer Science
Department of Mathematics
Baritiu 28, 400027 Cluj-Napoca, Romania
E-mail address: Iulia.Costin@cs.utcluj.ro
E-mail address: Gheorghe.Toader@math.utcluj.ro