On the rate of convergence of Archimedean double sequences

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ABSTRACT. Let $M$ and $N$ be two means. The pair of sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ defined by
\[ a_{n+1} = M(a_n, b_n), \quad b_{n+1} = N(a_{n+1}, b_n), \quad n \geq 0, \]
is called an Archimedean double sequence. We study the rate of convergence of these sequences to a common limit.

1. Introduction

There are more definitions of means. We use here the following one.

Definition 1. A mean (on the interval $J$) is defined as a function $M : J^2 \to J$, which has the property
\[ \min(a, b) \leq M(a, b) \leq \max(a, b), \quad \forall a, b \in J. \]
The mean $M$ is called symmetric if
\[ M(a, b) = M(b, a), \quad \forall a, b \in J. \]

We shall refer to weighted Gini means, defined by
\[ B_{r,s,\lambda}(a, b) = \left[ \frac{\lambda \cdot a^r + (1 - \lambda) \cdot b^r}{\lambda \cdot a^s + (1 - \lambda) \cdot b^s} \right]^{\frac{1}{r-s}}, \quad r \neq s \]
and
\[ B_{r,r,\lambda}(a, b) = \left[ a^{\lambda \cdot r} \frac{b(1-\lambda) \cdot b^r}{a^{1-\lambda} \cdot b^r} \right]^{\frac{1}{r-s}}, \quad r \neq s \]
with $\lambda \in [0, 1]$ fixed. They are symmetric only for $\lambda = 1/2$. If $s = 0$ we get the special case of weighted power means.

We shall use the following results regarding the partial derivatives of means and which can be found in [7].

Theorem 1. If $M$ is a differentiable mean then
\[ M_a(c, c) + M_b(c, c) = 1 \]
and
\[ 0 \leq M_a(c, c) \leq 1. \]

As a special case we get the next result proved in [4].
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Corollary 1. If $M$ is a symmetric differentiable mean then
$$M_a(c, c) = M_b(c, c) = \frac{1}{2}.$$  

Remark 1. For non symmetric means, this property is not valid. For example, for $M = B_{r,s;\lambda}$, we have
$$M_a(c, c) = \lambda.$$  

2. ARCHIMEDEAN DOUBLE SEQUENCES

The well known Archimedes’ polygonal method of evaluation of $\pi$, was interpreted in [5] as a double sequence. This led to the next definition. Let us consider two means $M$ and $N$ defined on the interval $J$ and two initial values $a, b \in J$.

Definition 2. The pair of sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ defined by
$$a_{n+1} = M(a_n, b_n) \quad \text{and} \quad b_{n+1} = N(a_{n+1}, b_n), \quad n \geq 0$$  
where $a_0 = a, b_0 = b$, is called an Archimedean double sequence.

Definition 3. The mean $M$ is composable in the sense of Archimedes (or $A$-composable) with the mean $N$ if the sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ defined by (2) are convergent to a common limit $M \boxtimes N(a, b)$ for each $a, b \in J$.

Remark 2. In this case $M \boxtimes N$ is also a mean on $J$ called Archimedean compound mean (or $A$-compound mean).

Remark 3. The classical case of Archimedes corresponds to the composition $H \boxtimes G$, where $H$ and $G$ denote the harmonic mean, respectively the geometric mean, defined by
$$H(a, b) = \frac{2ab}{a + b}, \quad G(a, b) = \sqrt{ab}, \quad \forall a, b > 0.$$  
As was determined in [5], if $0 < b_0 < a_0$, the common limit of the sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ is
$$H \boxtimes G(a_0, b_0) = \frac{a_0 b_0}{\sqrt{a_0^2 - b_0^2}} \arccos \frac{b_0}{a_0}.$$  
In Archimedes’ case, as
$$a_0 = 3\sqrt{3} \quad \text{and} \quad b_0 = 3\sqrt{3}/2,$$
the common limit is $\pi$.

3. RATE OF CONVERGENCE

In the case of classical Archimedean algorithm it is shown that the error of the sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ tend to zero asymptotically like $1/4^n$. In [4] is proved that this result is valid in the case of $A$-composition of arbitrary differentiable symmetric means. For the general case, we have the following evaluation.

For two means $M$ and $N$ given on the interval $J$ and two initial values $a, b \in J$ we denote
$$\alpha = M \boxtimes N(a, b).$$
Theorem 2. If the means $M$ and $N$ have continuous partial derivatives up to second order, then the errors of the sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ tend to zero asymptotically like

$$[M_a(\alpha, \alpha) \cdot (1 - N_a(\alpha, \alpha))]^n.$$  

Proof. If we write

$$a_n = \alpha + \delta_n, \ b_n = \alpha + \varepsilon_n,$$

we deduce that, as $n \to \infty$,

$$\alpha + \delta_{n+1} = M(\alpha + \delta_n, \alpha + \varepsilon_n)$$

$$= M(\alpha, \alpha) + M_a(\alpha, \alpha)\delta_n + M_b(\alpha, \alpha)\varepsilon_n + O(\delta_n^2 + \varepsilon_n^2).$$

From (1) we get

$$\delta_{n+1} = M_a(\alpha, \alpha)\delta_n + [1 - M_a(\alpha, \alpha)]\varepsilon_n + O(\delta_n^2 + \varepsilon_n^2)$$

(3)

Then

$$\alpha + \varepsilon_{n+1} = N(\alpha + \delta_{n+1}, \alpha + \varepsilon_n)$$

$$= N(\alpha, \alpha) + N_a(\alpha, \alpha)\delta_{n+1} + N_b(\alpha, \alpha)\varepsilon_n + O(\delta_{n+1}^2 + \varepsilon_n^2).$$

Using again (1) and (3) we have

$$\varepsilon_{n+1} = N_a(\alpha, \alpha)[M_a(\alpha, \alpha)\delta_n + (1 - M_a(\alpha, \alpha))\varepsilon_n]$$

$$+[1 - N_a(\alpha, \alpha)]\varepsilon_n + O(\delta_n^2 + \varepsilon_n^2),$$

thus

$$\varepsilon_{n+1} = M_a(\alpha, \alpha)N_a(\alpha, \alpha)\delta_n + [1 - M_a(\alpha, \alpha)]N_a(\alpha, \alpha)\varepsilon_n + O(\delta_n^2 + \varepsilon_n^2)$$

(4)

Subtracting (4) from (3) we get

$$\delta_{n+1} - \varepsilon_{n+1} = M_a(\alpha, \alpha)[1 - N_a(\alpha, \alpha)](\delta_n - \varepsilon_n) + O(\delta_n^2 + \varepsilon_n^2).$$

On the other hand, from the monotonicity of $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ we can suppose that $\delta_n > 0$ and $\varepsilon_n < 0$ for all $n > 0$. The case when $\delta_n < 0$ and $\varepsilon_n > 0$ can be treated similarly. We have

$$\frac{\varepsilon_n - \varepsilon_{n+1}}{\delta_n - \delta_{n+1}} = \frac{M_a(\alpha, \alpha)N_a(\alpha, \alpha)(\delta_n - \delta_n) + O(\delta_n^2 + \varepsilon_n^2)}{[1 - M_a(\alpha, \alpha)](\delta_n - \varepsilon_n) + O(\delta_n^2 + \varepsilon_n^2)},$$

thus

$$\varepsilon_n - \varepsilon_{n+1} = \frac{M_a(\alpha, \alpha)N_a(\alpha, \alpha)(\delta_n - \delta_{n+1}) + (\delta_n - \delta_{n+1})O(|\delta_n| + |\varepsilon_n|)}{M_a(\alpha, \alpha) - 1}.$$

Replacing $n$ by $n + 1, n + 2, ..., n + p - 1$ ($p \in \mathbb{N}$), adding and using the fact that $\delta_n$ and $\varepsilon_n$ tend monotonically to zero, we obtain

$$\varepsilon_n - \varepsilon_{n+p} = \frac{M_a(\alpha, \alpha)N_a(\alpha, \alpha)(\delta_n - \delta_{n+p}) + (\delta_n - \delta_{n+p})O(|\delta_n| + |\varepsilon_n|)}{M_a(\alpha, \alpha) - 1}.$$

Letting $p \to \infty$ we get

$$\varepsilon_n = \frac{M_a(\alpha, \alpha)N_a(\alpha, \alpha)}{M_a(\alpha, \alpha) - 1} \delta_n + O(\delta_n^2 + \varepsilon_n^2).$$

Using (3) we deduce that

$$\delta_{n+1} = M_a(\alpha, \alpha)[1 - N_a(\alpha, \alpha)]\delta_n + O(\delta_n^2).$$
and from (4) we have
\[ \varepsilon_{n+1} = M_a(\alpha, \alpha)[1 - N_a(\alpha, \alpha)] \varepsilon_n + O(\varepsilon_n^2). \]

\[ \square \]

Remark 4. In the case of symmetric means, we saw that
\[ M_a(\alpha, \alpha) = N_a(\alpha, \alpha) = \frac{1}{2}, \quad \forall \alpha \in J \]
and we get the result proved in [4]:

**Corollary 2.** If the means \( M \) and \( N \) are symmetric and have continuous partial derivatives up to second order, then the error of the sequences \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) tend to zero asymptotically like \(1/4^n\).

**Example 1.** For \( M = B_{r,s;\lambda} \) and \( N = B_{p,q;\mu} \), the sequences \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) tend to zero asymptotically like \([\lambda (1 - \mu)]^n\).

**Remark 5.** In [2] will be given a method of acceleration of the convergence, that is, it will be constructed a combination of the sequences which converges faster than each of them. For some symmetric means, such a method was used even in [6].

**References**


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