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On some positive matrices

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ABSTRACT. In this paper one gives elementary methods for solving some difficult problems related to any incidence matrix.

1. INTRODUCTION

This paper does not contain essential new theoretical results. Its aim is to suggest a method for solving some elementary problems, which are very difficult and are related to positive defined matrices. The best-known example of incidence matrices is the following: if $A_1, A_2, \ldots, A_n$ are finite sets, then the matrix $A = [a_{ij}]_{i,j=\overline{1,n}}$ (where $a_{ij} = |A_i \cap A_j|$ is the number of elements of the set $A_i \cap A_j$), is called the incident matrix of these sets.

One of the problems that require an ingenious solving, is to prove that the determinant of this matrix is positive. Starting from this problem, we are going to give a more general result.

Problem 1. *Prove that, if* $A_1, A_2, \ldots, A_n$ *are finite sets and* $a_{ij} = |A_i \cap A_j|$, *then* $\det[a_{ij}]_{i,j=\overline{1,n}} \geq 0$.

2. MAIN RESULTS

We begin with the solution of Problem 1, whose idea will be used to extend the method for another problems.

Proof. Consider $M = \bigcup\limits_{i=1}^{n} A_i = \{a_1, a_2, \ldots, a_N\}$.

To each set A_i we assign a vector $V_i \in \{0,1\}^N$, $V_i = (v_{i1}, v_{i2}, \ldots, v_{iN})$ where

$$v_{ik} = \begin{cases} 1 & \text{if} \quad a_k \in A_i \\ 0 & \text{if} \quad a_k \notin A_i \end{cases}$$

We have:

$$|A_i \cap A_j| = \sum_{k=1}^{N} v_{ij} v_{jk}, \quad i, j = \overline{1,n}.$$

So, considering the matrix $B \in \mathcal{M}_{m,N}(\{0,1\})$, $B = [v_{ik}]_{\substack{i=\overline{1,n} \\ k=\overline{1,N}}}$, we obtain that $A = B \cdot B^T$ (where B^T is the transpose of the matrix B).

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Lemma 1. *If $B \in \mathcal{M}_{n,N}(\mathbb{R})$ and $A = B \cdot B^T \in \mathcal{M}_n(\mathbb{R})$, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \geq 0,$$

for every $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Proof.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{N} v_{ik}v_{kj}x_i x_j =$$

$$= \sum_{k=1}^{N}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i v_{ik})(x_j v_{jk})\right) = \sum_{k=1}^{N}\left(\sum_{i=1}^{N} x_i v_{ik}\right)^2 \geq 0.$$

Lemma 2. *If $A \in \mathcal{M}_n(\mathbb{R})$ and $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \geq 0$ for every $x_1, x_2, \ldots, x_n \in \mathbb{R}$, then* $\det A \geq 0$.

Proof. We prove that $\det(A + xI_n) \geq 0$ for every $x \geq 0$.

In order to do this, it is enough to prove that the real polynomial $P(X) = \det(A + xI_n)$ has no positive roots.

If there exists $x_0 \in \mathbb{R}$ such that $\det(A + x_0 I_n) = 0$, then the system $(A + x_0 I_n)X = 0$ has a nontrivial solution $X_0 = [x_1, \ldots, x_n]^T$ where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then from $AX_0 = -x_0 X_0$ it follows that $X_0^T A X_0 = -x_0 X_0^T X_0$, which is equivalent to

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = -x_0 \sum_{i=1}^{n} x_i^2.$$

Hence, $x_0 \leq 0$.

So the polynomial P is positive on the interval $[0, \infty)$, whence $P(0) = \det A \geq 0$.

We conclude that Lemma 1 and Lemma 2 imply $\det A = \det(B \cdot B^T) \geq 0$ and this finishes the proof.

Remark 1. Considering the Euclidean scalar product on $\mathbb{R}^N$

$$\langle V_i, V_j \rangle = \sum_{k=1}^{N} v_{ik}v_{jk},$$

with we obtain that a_{ij} defined in (1) has the entries $a_{ij} = \langle V_i, V_j \rangle$, so the matrix A is the Gram matrix of the vectors $V_1, V_2, \ldots, V_n$, which will be denoted by $A = G[V_1, V_2, \ldots, V_n]$.

It is known that every Gram matrix is positive. More precisely for every Gram matrix $G = [\langle V_i, V_j \rangle]_{i,j=\overline{1,n}}$ we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \langle V_i, V_j \rangle x_i x_j = \sum_{i=1}^{n} \langle x_i V_i, x_j V_j \rangle =$$

$$= \left\langle \sum_{i=1}^{n} x_i V_i, \sum_{j=1}^{n} x_j V_j \right\rangle = \left\| \sum_{i=1}^{n} x_i V_i \right\|^2 \geq 0,$$

hence, using Lemma 2, $\det(G) \geq 0$.

Remark 2. If we consider the characteristic functions of the sets A_i,

$$\varphi_{A_i} : \bigcup_{i=1}^{n} A_i \to \{0,1\}, \quad \varphi_{A_i}(x) = \begin{cases} 1 & \text{if} \quad x \in A_i \\ 0 & \text{if} \quad x \notin A_i \end{cases}$$

and on the set of functions $\{\varphi : \{a_1, a_2, \ldots, a_N\} \to \{0,1\}\}$ we define the scalar product

$$\langle \varphi_1, \varphi_2 \rangle = \sum_{k=1}^{N} \varphi_1(a_k)\varphi_2(a_k),$$

then

$$a_{ij} = |A_i \cap A_j| = \langle \varphi_{A_i}, \varphi_{A_j} \rangle.$$

Hence the matrix A can also be written as a Gram matrix of characteristic functions:

$$A = G[\varphi_{A_1}, \varphi_{A_2}, \ldots, \varphi_{A_n}].$$

Further, we extend these results. First we recall some definitions.

Definition 1. A family of sets K is called a **clan of sets** if for every A, B in K we have $A \cup B \in K$ and $A - B \in K$.

Remark 3. If K is a clan of sets, then $\emptyset \in K$ and $A \cap B \in K$ for every A and B in K.

Definition 2. If K is a clan of sets, then a function $m : K \to [0, \infty]$ is called a **measure** on K if $m(A \cup B) = m(A) + m(B)$, for every disjoint sets $A, B \in K$.

Remark 4. If K is a clan endowed with a measure m, denoted (K, m) then:

a) $m(\emptyset) = 0$

b) If $A \subset B$ then $m(B - A) = m(B) - m(A)$

c) If $m(A \cup B) \in [0, \infty)$, then $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.

Definition 3. If (K, m) is a clan endowed with a measure and $A_1, A_2, \ldots, A_n$ are sets from K, then the matrix $A = [a_{ij}]_{i,j=\overline{1,n}}$, where $a_{ij} = m(A_i \cap A_j)$ is called the **incident matrix** of the sets $A_1, A_2, \ldots, A_n$.

A generalization of the Problem 1 is:

Problem 2. Prove that if $A = [m(A_i \cap A_j)]_{i,j=\overline{1,n}}$ is the incident matrix of the sets $A_1, A_2, \ldots, A_n$, then the determinant of the matrix is positive.

Proof. We prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \geq 0$$

for every real numbers $x_1, x_2, \ldots, x_n$ and from Lemma 2 it follows that $\det A \geq 0$. Denoting $I = \{1, 2, \ldots, n\}$, for every nonempty subset J of I we define the sets

$$B_J = \bigcap_{j \in J} A_j - \bigcup_{j \notin J} A_i.$$

The $2^n - 1$ sets B_J, with $J \subset I$,form a partition of the set $B = \bigcup_{i=1}^{n} A_i$ and we have

$$A_i = \bigcup_{J} \{B_J |\ i \in J\} \text{ and } A_i \cap A_j = \bigcup_{J} \{B_J |\ \{i,j\} \subset J\}.$$

Denoting $m(B_J) = m_J$, from the properties of the measure and from the above relations we obtain:

$$m(A_i) = \sum_{i\in J} m_J = a_{ii} \text{ and } m(A_i \cap A_j) = \sum_{\{i,j\}\subset J} m_J = a_{ij}.$$

Then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{\{i,j\}\subset J} m_J\right) x_i x_j =$$

$$= \sum_J m_J \left(\sum_{i\in J}\sum_{j\in J} x_i x_j\right) = \sum_J m_j \left(\sum_{j\in J} x_i\right)^2 \geq 0,$$

From Lemma 1 and Lemma 2 we conclude that $detA \geq 0$.

Remark 5. If we denote the $2^n - 1 = N$ nonempty subsets of I by $J_1, J_2, \ldots, J_N$ and if we assign to every set A_i a vector $V_i = (v_{i1}, v_{i2}, \ldots, v_{iN}) \in \mathbb{R}^n$, where

$$v_{ik} = \begin{cases} \sqrt{m_{J_k}} & \text{if } i \in J_k, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\langle V_i, V_j \rangle = \sum_{k=1}^{N} v_{ik}v_{jk} = \sum_{\{i,j\}\in J_k} m_{J_k} = a_{ij}.$$

So the matrix A is a Gram matrix:

$$A = G[V_1, V_2, \ldots, V_n],$$

or $A = B \cdot B^T$, where $B \in \mathcal{M}_{n,N}(\mathbb{R})$ is the matrix which has the rows $L_i = [v_{i1}, v_{i2}, \ldots, v_{iN}]$, $i = \overline{1,n}$.

In many particular cases, Problem 2 admits different solutions.

Problem 3. We consider $D_1, D_2, \ldots, D_n$ a set of discs on the plane (Jordan measurable sets) and we denote the area of the set $D_i \cap D_j$ by $S(D_i \cap D_j)$. Prove that for every real numbers $x_1, x_2, \ldots, x_n$ the following relation holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} S(D_i \cap D_j)x_i x_j \geq 0.$$

Proof. If we consider the clan of the Jordan measurable sets from the plane and take as measure of such a set its area, then Problem 3 is a particular case of Problem 2. In what follows, we give another solution of this problem, independent of the previous mentioned one.

Considering $\varphi_{D_i} : \mathbb{R}^2 \to \{0, 1\}$ the characteristic function of the set $D_i \subseteq \mathbb{R}^2$,

$$\varphi_{D_i}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in D_i, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\varphi_{D_i \cap D_j} = \varphi_{D_i} \cdot \varphi_{D_j},$$

$$S(D_i) = \iint_{\mathbb{R}^2} \varphi_{D_i}(x, y)dxdy = \iint_{\mathbb{R}^2} (\varphi_{D_i}(x, y))^2 dxdy$$

and

$$S(D_i \cap D_j) = \iint_{\mathbb{R}^2} \varphi_{D_i \cap D_j}(x,y)dxdy = \iint_{\mathbb{R}^2} \varphi_{D_i}(x,y) \cdot \varphi_{D_j}(x,y)dxdy.$$

Then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} S(D_i \cap D_j)x_i x_j = \iint_{\mathbb{R}^2} \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \cdot \varphi_{D_i}(x,y) \cdot x_j \cdot \varphi_{D_j}(x,y)dxdy =$$

$$= \iint_{\mathbb{R}^2} (x_1 \cdot \varphi_{D_1}(x,y) + \cdots + x_n \cdot \varphi_{D_n}(x,y))^2 dxdy \geq 0.$$

Remark 6. The matrix $A = [a_{ij}]_{i,j=\overline{1,n}}$, where $a_{ij} = S(D_i \cap D_j)$ is the Gram matrix of the functions $\varphi_{D_1}, \ldots, \varphi_{D_n} \in L^2(\mathbb{R}^2)$.

In what follows we give two problems related to the number theory.

Problem 4. Let us consider $n \in \mathbb{N}^*$, $a_i \in \mathbb{N}^*$, $b_i \in \mathbb{Z}$, $i = 1, \ldots, n$ be given. We denote by $d_{ij} = (a_i, a_j)$ the greatest common divisor of a_i and a_j and define $p_{ij} = b_i \cdot b_j$. Prove that

$$\prod_{i=1}^{n}\prod_{j=1}^{n} d_{ij}^{p_{ij}} \geq 1.$$

Proof. Taking the logarithm we obtain

$$\prod_{i,j=1}^{n} d_{ij}^{p_{ij}} \geq 1 \ \Leftrightarrow\ \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} \log d_{ij} \geq 0.$$

Let $M = [a_1, a_2, \ldots, a_n]$ be the least common multiple of the numbers $a_1, a_2, \ldots, a_n$ and $M = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$ its decomposition in prime factors.

To each number $a_i = p_1^{\alpha_{i1}} p_2^{\alpha_{i2}} \ldots p_k^{\alpha_{ik}}$ we assign a vector $V_i \in \mathbb{R}^m$, where $m = m_1 + m_2 + \cdots + m_k$, in the following way.

$$V_i = [\underbrace{\sqrt{\log p_1}, \ldots, \sqrt{\log p_1}}_{\alpha_{i1}}, \underbrace{0, \ldots, 0}_{m_1 - \alpha_{i1}}, \ldots, \underbrace{\sqrt{\log p_k}, \ldots, \sqrt{\log p_k}}_{\alpha_{ik}}, \underbrace{0, \ldots, 0}_{m_k - \alpha_{ik}}] =$$

$$= [v_{i1}, \ldots, v_{im}]$$

Therefore we have

$$\log d_{ij} = \sum_{q=1}^{m} v_{iq} \cdot v_{jq},$$

so

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} \log d_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i \cdot b_j \sum_{q=1}^{m} v_{iq} \cdot v_{jq} =$$

$$= \sum_{q=1}^{m} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} b_i \cdot v_{iq} \cdot b_j \cdot v_{jq} \right) = \sum_{q=1}^{m} \left(\left(\sum_{i=1}^{n} b_i \cdot v_{iq} \right) \left(\sum_{j=1}^{n} b_j \cdot v_{jq} \right) \right) =$$

$$= \sum_{q=1}^{m} \left(\sum_{i=1}^{n} b_i \cdot v_{iq} \right)^2 \geq 0.$$

Problem 5. We consider $n \in \mathbb{N}^*$, $a_i \in \mathbb{N}^*$, $b_i \in \mathbb{Z}$, $i = 1, \ldots, n$, and (a_i, a_j) the greatest common divisor of the numbers a_i and a_j. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i, a_j) b_i \cdot b_j \geq 0.$$

Proof. If p and q are natural numbers, then the set $U_p = \{z \in \mathbb{C} |\ z^p = 1\}$ has p elements, the set $\{U_q = \{z \in \mathbb{C} |\ z^q = 1\}$ has q elements and the set $U_p \cap U_q = \{z \in \mathbb{C} |\ z^p = z^q = 1\}$ has $d = (p, q)$ elements $(z^p = z^q = 1 \Longleftrightarrow z^d = 1)$.

Then $(a_i, a_j) = |U_{a_i} \cap U_{a_j}|$ is the number of elements of the set $U_{a_i} \cap U_{a_j}$ and thus this Problem reduces to Problem 1 (to Lemma 1 and Lemma 2).

Problem 6. We consider $A = [a_{ij}]_{i,j=\overline{1,n}}$, where $a_{ij} = \dfrac{n}{[i, j]}$, with $[i, j]$ the least common divisor of i and j. Prove that $\det A = 1$.

Proof. The integer part of the number n/m is equal to the number of all multiples of m which are less than or equal to n. If M_m is the set of those multiples of m which are less than or equal to n, then $a_{ij} = |M_i \cap M_j|$. Thus, by Problem 1, the matrix A is a Gram matrix. Defining the matrix B as

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \ldots & \ldots & \ldots & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & \ldots & \ldots & \ldots & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & \ldots & \ldots & \ldots & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 1 \end{bmatrix},$$

then $A = B \cdot B^T$ and $\det A = (\det B)^2 = 1$.

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