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Symmetries with Greek means

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Abstract. A mean N is called complementary to M with respect to P if it verifies the relation

$$P(M(a,b),N(a,b))=P(a,b),\forall a,b>0.$$

We look for the complementary of a Greek mean with respect to another. We determine all the cases in which it is again a Greek mean.

1. Introduction

As a brief definition of means, usually is given the following

Definition 1. A **mean** is a function $M:\mathbb{R}_+^2\to\mathbb{R}_+$, which has the property

$$a\wedge b\le M(a,b)\le a\vee b,\ \forall a,b>0$$

where

$$a\wedge b=\min(a,b) \text{ and } a\vee b=\max(a,b).$$

We remark that $\wedge$ and $\vee$ are means.

We use ordinary notations for operations with functions. For example $M+N$ is defined by

$$(M+N)(a,b)=M(a,b)+N(a,b),\ \forall a,b>0.$$

Of course, if M and N are means, the result of the operation with functions is not a mean. We have to combine more operations with functions to get a (partial) operation with means. Such a method is the **composition** of means. Given three means M,N and P , the expression

$$P(M,N)(a,b)=P(M(a,b),N(a,b)),\ \forall a,b>0,$$

defines also a mean $P(M,N)$.

We write also

$$M\le N$$

to denote

$$M(a,b)\le N(a,b),\ \forall a,b>0.$$

Using the method of proportions, the Pythagorean school defined ten means: the arithmetic mean $\mathcal{A}$, the geometric mean $\mathcal{G}$ the harmonic mean $\mathcal{H}$ the contraharmonic mean $\mathcal{C}$ and six unnamed means $\mathcal{F}_i,\ i=5,...,10$. As many other important Greek mathematical contributions, the means are presented by Pappus of Alexandria in his books, in the fourth century AD (see [5]). Some indications about them can be found in the books [4, 2, 1, 3], or in [7].

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Using operations with $\wedge$ and $\vee$, we can give the Greek means as follows:

$$\mathcal{A}=\frac{\vee+\wedge}{2},\ \mathcal{G}=\sqrt{\vee\wedge},\ \mathcal{H}=\frac{2\vee\wedge}{\vee+\wedge},\ \mathcal{C}=\frac{\vee^2+\wedge^2}{\vee+\wedge},$$

$$\mathcal{F}_5=\frac{1}{2}\left[\vee-\wedge+\sqrt{(\vee-\wedge)^2+4\wedge^2}\right],\ \mathcal{F}_6=\frac{1}{2}\left[\wedge-\vee+\sqrt{(\vee-\wedge)^2+4\vee^2}\right],$$

$$\mathcal{F}_7=\frac{\vee^2-\vee\wedge+\wedge^2}{\vee},\ \mathcal{F}_8=\frac{\vee^2}{2\vee-\wedge},$$

$$\mathcal{F}_9=\frac{\wedge(2\vee-\wedge)}{\vee}\text{ and }\mathcal{F}_{10}=\frac{1}{2}\left[\wedge+\sqrt{\wedge(4\vee-3\wedge)}\right].$$

By the same method the power means (or Hölder means) are given by:

$$\mathcal{P}_n=\left(\frac{\wedge^n+\vee^n}{2}\right)^{\frac{1}{n}},\ n\neq 0$$

and the n - counter-harmonic mean (or Lehmer mean):

$$\mathcal{C}_n=\frac{\wedge^n+\vee^n}{\wedge^{n-1}+\vee^{n-1}}.$$

2. Complementary means

In [6] is given the following definition which is then used for the case of the Greek means.

Definition 2. A mean N is called **complementary to** M **with respect to** P (or $P-$**complementary** to M) if it verifies

$$P(M,N)=P$$

Then in [8] was determined the complementaries of the Greek means with respect to the first five Greek means. In this paper we want to continue by determining the complementaries of the Greek means with respect to the last five Greek means. We shall denote the complementary of the mean M with respect to $\mathcal{F}_i$ by $M^{\mathcal{F}i}$. In [6] was given the following

Theorem 1. *We have successively*

$$M^{\mathcal{F}6}=\begin{cases}\mathcal{F}_6+M-\frac{M^2}{\mathcal{F}_6} & ,\ if\ \mathcal{F}_6\leq M\\ \frac{1}{2}\left[\mathcal{F}_6+\sqrt{\mathcal{F}_6(5\mathcal{F}_6-4M)}\right] & ,\ if\ \mathcal{F}_6\geq M\end{cases};$$

$$M^{\mathcal{F}7}=\begin{cases}\frac{1}{2}\left[M+\sqrt{M(4\mathcal{F}_7-3M)}\right] & ,\ if\ \mathcal{F}_7\leq M\\ \frac{1}{2}\left[M+\mathcal{F}_7+\sqrt{\mathcal{F}_7^2+2M\mathcal{F}_7-3M^2}\right] & ,\ if\ \mathcal{F}_7\geq M\end{cases};$$

$$M^{\mathcal{F}8}=\begin{cases}2M-\frac{M^2}{\mathcal{F}_8} & ,\ if\ \mathcal{F}_8\leq M\\ \mathcal{F}_8+\sqrt{\mathcal{F}_8(\mathcal{F}_8-M)} & ,\ if\ \mathcal{F}_8\geq M\end{cases};$$

$$M^{\mathcal{F}9}=\begin{cases} M-\sqrt{M(M-\mathcal{F}_9)}\,, & \textit{if } \mathcal{F}_9\le M \\ \frac{M^2}{2M-\mathcal{F}_9} & , \textit{if } \mathcal{F}_9\ge M \end{cases};$$

$$M^{\mathcal{F}10}=\begin{cases} \frac{1}{2}\left[M+\mathcal{F}_{10}-\sqrt{M^2+2M\mathcal{F}_{10}-3\mathcal{F}_{10}^2}\right], & \textit{if } \mathcal{F}_{10}\le M \\ M-\mathcal{F}_{10}+\frac{\mathcal{F}_{10}^2}{M} & , \textit{if } \mathcal{F}_{10}\ge M \end{cases}.$$

Using it, we can determine later the complementaries:

Theorem 2. *Let $t2>1$ respectively $t4>1$ be the roots of the equations*

$$t^3-t^2-2t+1=0,\ t^3-3t^2+2t-1=0\,.$$

The complementary of the Greek means with respect to $\mathcal{F}_6$ are:

$$\mathcal{A}^{\mathcal{F}\,6}=\frac{1}{4}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge+\sqrt{2}\right.$$

$$\left.\cdot\sqrt{17\vee^2-10\vee\wedge+3\wedge^2-(7\vee-3\wedge)\sqrt{(\vee-\wedge)^2+4\vee^2}}\right];$$

$$\mathcal{G}^{\mathcal{F}\,6}=\frac{1}{4}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge+\sqrt{2}\right.$$

$$\left.\cdot\sqrt{15\vee^2-10\vee\wedge+5\wedge^2+4\sqrt{\vee\wedge}\,(\vee-\wedge)-\left(5\vee-5\wedge+4\sqrt{\vee\wedge}\right)\sqrt{(\vee-\wedge)^2+4\vee^2}}\right];$$

$$\mathcal{H}^{\mathcal{F}\,6}=\frac{1}{4}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge+\sqrt{\frac{2}{\vee+\wedge}}\right.$$

$$\left.\cdot\sqrt{15\vee^3+13\vee^2\wedge-13\vee\wedge^2+5\wedge^3-(5\vee^2+8\vee\wedge-5\wedge^2)\sqrt{(\vee-\wedge)^2+4\vee^2}}\right];$$

$$\mathcal{C}^{\mathcal{F}\,6}=\frac{\wedge}{2\vee^2(\vee+\wedge)^2}\left[2\vee^4+\vee^3\wedge+5\vee^2\wedge^2-\vee\wedge^3+\wedge^4\right.$$

$$\left.+\left(2\vee^3-\vee^2\wedge-\wedge^3\right)\sqrt{(\vee-\wedge)^2+4\vee^2}\right];$$

$$\mathcal{F}_5^{\mathcal{F}\,6}=2\wedge+\frac{1}{4\vee^2}$$

$$\cdot\left[\sqrt{(\vee-\wedge)^2+4\wedge^2}-\vee-3\wedge\right]\left[\vee^2+2\vee\wedge-\wedge^2+(\wedge-\vee)\sqrt{(\vee-\wedge)^2+4\vee^2}\right]$$

$$\mathcal{F}_7^{\mathcal{F}\,6}=$$

$$=\begin{cases} \left[38\vee^3-36\vee^2\wedge+26\vee\wedge^2-3\wedge^3-\left(18\vee^2-18\vee\wedge+8\wedge^2\right)\right. \\ \left.\cdot\sqrt{(\vee-\wedge)^2+4\vee^2}\right]^{1/2}\cdot\frac{1}{2\sqrt{\vee}}+\frac{1}{2}\left(\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge\right), & \vee\le t4\cdot\wedge \\ \frac{1}{4\vee^4}\left[3\vee^5-5\vee^4\wedge+9\vee^3\wedge^2-5\vee^2\wedge^3+3\vee\wedge^4-\wedge^5\right. \\ \left.+\left(\vee^4+2\vee^3\wedge-3\vee^2\wedge^2+2\vee\wedge^3-\wedge^4\right)\sqrt{(\vee-\wedge)^2+4\vee^2}\right], & \vee\ge t4\cdot\wedge \end{cases};$$

$$\mathcal{F}_8^{\mathcal{F}\,6}=\frac{1}{2}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge+\sqrt{\frac{2}{2\vee-\wedge}}\right.$$

$$\left.\cdot\sqrt{34\vee^3-39\vee^2\wedge+20\vee\wedge^2-5\wedge^3-(14\vee^2-15\vee\wedge+5\wedge^2)\sqrt{(\vee-\wedge)^2+4\vee^2}}\right];$$

$$\mathcal{F}_9^{\mathcal{F}\,6}=$$

$$=\begin{cases}\frac{1}{2\vee^4}\left[\left(\vee^4-4\vee^2\wedge^2+4\vee\wedge^3-\wedge^4\right)\sqrt{(\vee-\wedge)^2+4\vee^2}\right.\\ \left.-\vee^5+5\vee^4\wedge-6\vee^3\wedge^2+8\vee^2\wedge^3-5\vee\wedge^4+\wedge^5\right],\vee\le t2\cdot\wedge\\ \sqrt{15\vee^3-2\vee^2\wedge-7\vee\wedge^2+4\wedge^3-(5\vee^2+3\vee\wedge-4\wedge^2)\sqrt{(\vee-\wedge)^2+4\vee^2}}\\ \cdot\sqrt{\frac{1}{8\vee}}+\frac{1}{4}\left(\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge\right),\ \vee\ge t2\cdot\wedge\end{cases};$$

$$\mathcal{F}_{10}^{\mathcal{F}\,6}=$$

$$=\begin{cases}\frac{1}{4\vee^2}\left[\left(2\vee^2-2\vee\wedge+\wedge^2\right)\sqrt{(\vee-\wedge)^2+4\vee^2}+\left(2\vee^2-\vee\wedge+\wedge^2\right)\right.\\ \cdot\sqrt{\wedge(4\vee-3\wedge)}-\wedge\sqrt{\wedge(4\vee-3\wedge)}\cdot\sqrt{(\vee-\wedge)^2+4\vee^2}-2\vee^3\\ \left.+2\vee^2\wedge+3\vee\wedge^2-\wedge^3\right],\ \vee\le t4\cdot\wedge\\ \sqrt{10\vee^2-\left(\sqrt{(\vee-\wedge)^2+4\vee^2}+\wedge-\vee\right)\cdot\left(2\sqrt{\wedge(4\vee-3\wedge)}+5\vee-3\wedge\right)}\\ \cdot\frac{\sqrt{2}}{4}+\frac{1}{4}\left(\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge\right),\ \vee\ge t4\cdot\wedge\end{cases}$$

Theorem 3. *Let* $s1=\left(1+\sqrt{5}\right)/2$, *and* $t1>1$ *respectively* $t4>1$ *be the roots of the equations*

$$t^3-2t^2+t-1=0,\ t^3-3t^2+2t-1=0.$$

The complementary of the Greek means with respect to $\mathcal{F}_7$ *are:*

$$\mathcal{A}^{\mathcal{F}\,7}=\begin{cases}\frac{1}{4}\left[\vee+\wedge+\frac{1}{\sqrt{\vee}}\sqrt{(\vee+\wedge)(5\vee^2-11\vee\wedge+8\wedge^2)}\right],\vee\le 2\wedge\\ \frac{1}{4\vee}\left[3\vee^2-\vee\wedge+2\wedge^2+\sqrt{5\vee^4-14\vee^3\wedge+9\vee^2\wedge^2-4\vee\wedge^3+4\wedge^4}\right],\\ \vee\ge 2\wedge;\end{cases}$$

$$\mathcal{G}^{\mathcal{F}\,7}=\begin{cases}\frac{1}{2\vee}\left[\sqrt{\vee^4-5\vee^3\wedge+3\vee^2\wedge^2-2\vee\wedge^3+\wedge^4+2\vee\sqrt{\vee\wedge}(\vee^2-\vee\wedge+\wedge^2)}\right.\\ \left.+\vee^2-\vee\wedge+\wedge^2+\vee\sqrt{\vee\wedge}\right],\ \vee\le t1\cdot\wedge\\ \frac{1}{2}\left[\sqrt{\vee\wedge}+\sqrt{4\sqrt{\frac{\wedge}{\vee}}(\vee^2-\vee\wedge+\wedge^2)-3\vee\wedge}\right],\vee\ge t1\cdot\wedge\end{cases};$$

$$\mathcal{H}^{\mathcal{F}\,7}=\begin{cases}\frac{1}{\vee+\wedge}\left[\vee\wedge+\sqrt{\wedge(2\vee^3-3\vee^2\wedge+2\wedge^3)}\right],\vee\le s1\cdot\wedge\\ \frac{1}{2\vee(\vee+\wedge)}\left[\vee^3+2\vee^2\wedge+\wedge^3\right.\\ \left.+\sqrt{\vee^6+4\vee^5\wedge-12\vee^4\wedge^2+2\vee^3\wedge^3+4\vee^2\wedge^4+\wedge^6}\right],\vee\ge s1\cdot\wedge\end{cases}$$

$$\mathcal{C}^{\mathcal{F}\,7}=\frac{1}{2\vee(\vee+\wedge)}\left[\vee\left(\vee^2+\wedge^2\right)+\sqrt{\vee(\vee^2+\wedge^2)(\vee^3-3\vee\wedge^2+4\wedge^3)}\right];$$

$$\mathcal{F}_5^{\mathcal{F}\,7} = \frac{1}{4}\left[\sqrt{(\vee - \wedge)^2 + 4\wedge^2} + \vee - \wedge + \sqrt{\frac{2}{\vee}}\right.$$

$$\left.\cdot\sqrt{(\vee^2 - \vee\wedge + 4\wedge^2)\sqrt{(\vee - \wedge)^2 + 4\wedge^2} + \vee\,(\vee^2 - 2\vee\wedge - \wedge^2)}\right] \;;$$

$$\mathcal{F}_6^{\mathcal{F}\,7} =$$

$$= \begin{cases} \sqrt{\frac{1}{8\vee}}\left[\left(7\vee^2 - 7\vee\wedge + 4\wedge^2\right)\sqrt{(\vee - \wedge)^2 + 4\vee^2} - 13\vee^3 + 14\vee^2\wedge\right. \\ \left. - 11\vee\wedge^2 + 4\wedge^3\right]^{1/2} + \frac{1}{4}\left(\sqrt{(\vee - \wedge)^2 + 4\vee^2} - \vee + \wedge\right) \;,\; \vee \le t4\cdot\wedge \\ \frac{\sqrt{2}}{4\vee}\left[\vee\left(5\vee^2 - 5\vee\wedge + 2\wedge^2\right)\sqrt{(\vee - \wedge)^2 + 4\vee^2} - 9\vee^4 + 6\vee^3\wedge - \vee^2\wedge^2\right. \\ \left. - 2\vee\wedge^3 + 2\wedge^4\right]^{1/2} + \frac{1}{4\vee}\left(\vee\sqrt{(\vee - \wedge)^2 + 4\vee^2} + \vee^2 - \vee\wedge + 2\wedge^2\right) \;,\; \vee \ge t4\cdot\wedge \end{cases}$$

$$\mathcal{F}_8^{\mathcal{F}\,7} = \frac{1}{2\vee(2\vee - \wedge)}\left[3\vee^3 - 3\vee^2\wedge + 3\vee\wedge^2 - \wedge^3\right.$$

$$\left. + \sqrt{5\vee^6 - 18\vee^5\wedge + 27\vee^4\wedge^2 - 24\vee^3\wedge^3 + 15\vee^2\wedge^4 - 6\vee\wedge^5 + \wedge^6}\right] \;;$$

$$\mathcal{F}_9^{\mathcal{F}\,7} = \begin{cases} \frac{1}{2\vee}\left[\wedge(2\vee - \wedge) + \sqrt{\wedge(2\vee - \wedge)(4\vee^2 - 10\vee\wedge + 7\wedge^2)}\right] \;,\; \vee \le 2\wedge \\ \frac{\vee + \wedge}{2} + \frac{1}{2\vee}\sqrt{\vee^4 + 2\vee^3\wedge - 15\vee^2\wedge^2 + 16\vee\wedge^3 - 4\wedge^4} \;,\; \vee \ge 2\wedge \end{cases}$$

$$\mathcal{F}_{10}^{\mathcal{F}\,7} =$$

$$= \begin{cases} \sqrt{\frac{1}{8\vee}}\left[\left(4\vee^2 - 7\vee\wedge + 4\wedge^2\right)\sqrt{\wedge(4\vee - 3\wedge)}\right. \\ \left. - \wedge\left(2\vee^2 + \vee\wedge - 4\wedge^2\right)\right]^{1/2} + \frac{1}{4}\left(\sqrt{\wedge(4\vee - 3\wedge)} + \wedge\right) \;,\; \vee \le t4\cdot\wedge \\ \frac{\sqrt{2}}{4\vee}\left[2\vee^4 - 8\vee^3\wedge + 7\vee^2\wedge^2 - 2\vee\wedge^3 + 2\wedge^4 + \vee\left(2\vee^2 - 5\vee\wedge + 2\wedge^2\right)\right. \\ \left.\cdot\sqrt{\wedge(4\vee - 3\wedge)}\right]^{1/2} + \frac{1}{4\vee}\left[2\vee^2 - \vee\wedge + 2\wedge^2 + \vee\sqrt{\wedge(4\vee - 3\wedge)}\right] \;,\; \vee \ge t4\cdot\wedge \end{cases} .$$

Theorem 4. *Let* $s3 = \left(3 + \sqrt{5}\right)/2$, *and* $t5 > 1$ *the root of the equation*

$$t^3 - 5t^2 + 4t - 1 = 0 \,.$$

The complementary of the Greek means with respect to $\mathcal{F}_8$ *are:*

$$\mathcal{A}^{\mathcal{F}\,8} = \frac{1}{4\vee^2}\left(2\vee^3 + \vee^2\wedge + \wedge^3\right) \;;$$

$$\mathcal{G}^{\mathcal{F}\,8} = \begin{cases} 2\sqrt{\vee\wedge} - \frac{\wedge(2\vee - \wedge)}{\vee},\; \vee \le s3\cdot\wedge \\ \frac{\vee}{2\vee - \wedge}\left[\vee + \sqrt{\vee^2 - \sqrt{\vee\wedge}\,(2\vee - \wedge)}\right] \;,\; \vee \ge s3\cdot\wedge \end{cases}$$

$$\mathcal{H}^{\mathcal{F}\,8} = \begin{cases} \frac{4\wedge}{(\vee + \wedge)^2}\cdot\left(\vee^2 - \vee\wedge + \wedge^2\right) \;,\; \vee \le 2\wedge \\ \frac{\vee}{2\vee - \wedge}\left[\vee + \sqrt{\frac{\vee}{\vee + \wedge}}\cdot\sqrt{\vee^2 - 3\vee\wedge + 2\wedge^2}\right] \;,\; \vee \ge 2\wedge \end{cases} ;$$

$$\mathcal{C}^{\mathcal{F}\,8} = \frac{\wedge\left(\vee^2 + \wedge^2\right)}{\vee^2(\vee + \wedge)^2}\left(3\vee^2 - 2\vee\wedge + \wedge^2\right) \;;$$

$$\mathcal{F}_5^{\mathcal{F}\,8} = \frac{\wedge}{2\vee^2}\left[3\vee^2 - 8\vee\wedge + 3\wedge^2 + (3\vee - \wedge)\sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right] \;;$$

$$\mathcal{F}_6^{\mathcal{F}\,8} = \frac{1}{2\vee^2}\left[\left(4\vee^2 - 3\vee\wedge + \wedge^2\right)\sqrt{\left(\vee - \wedge\right)^2 + 4\vee^2} - 8\vee^3 + 9\vee^2\wedge - 4\vee\wedge^2 + \wedge^3\right] ;$$

$$\mathcal{F}_7^{\mathcal{F}\,8} = \frac{\wedge}{\vee^4}\left(\vee^2 - \vee\wedge + \wedge^2\right)\left(3\vee^2 - 3\vee\wedge + \wedge^2\right) ;$$

$$\mathcal{F}_9^{\mathcal{F}\,8} = \begin{cases} \frac{\wedge(2\vee-\wedge)}{\vee^4}\left(2\vee^3 - 4\vee^2\wedge + 4\vee\wedge^2 - \wedge^3\right), & \vee \le s3\cdot\wedge \\ \frac{\wedge}{(2\vee-\wedge)\sqrt{\vee}}\left[\vee\sqrt{\vee} + \sqrt{\left(\vee-\wedge\right)\left(\vee^2 - 3\vee\wedge + \wedge^2\right)}\right], & \vee \ge s3\cdot\wedge \end{cases} ;$$

$$\mathcal{F}_{10}^{\mathcal{F}\,8} = \begin{cases} \frac{\sqrt{\wedge}}{2\vee^2}\left[\left(2\vee^2 - 2\vee\wedge + \wedge^2\right)\sqrt{4\vee - 3\wedge} - \left(2\vee^2 - 4\vee\wedge + \wedge^2\right)\sqrt{\wedge}\right], \\ \qquad \vee \le t5\cdot\wedge \\ \frac{\vee}{2\vee-\wedge}\left[\vee + \frac{1}{\sqrt{2}}\sqrt{2\vee^2 - 2\vee\wedge + \wedge^2 - \left(2\vee - \wedge\right)\sqrt{\wedge\left(4\vee - 3\wedge\right)}}\right], \\ \qquad \vee \ge t5\cdot\wedge. \end{cases}$$

Theorem 5. *Let* $s1 = \left(1+\sqrt{5}\right)/2$, $s2 = 1+\sqrt{2}/2$ *and* $s3 = \left(3+\sqrt{5}\right)/2$ *, while* $t2 > 1$ *be the root of the equation*

$$t^3 - t^2 - 2t + 1 = 0 .$$

The complementary of the Greek means with respect to $\mathcal{F}_9$ *are:*

$$\mathcal{A}^{\mathcal{F}\,9} = \begin{cases} \frac{\vee(\vee+\wedge)^2}{4(\vee^2-\vee\wedge+\wedge^2)}, & \vee \le 2\wedge \\ \frac{1}{2}\left[\vee + \wedge - \sqrt{\frac{\vee+\wedge}{\vee}\left(\vee^2 - 3\vee\wedge + 2\wedge^2\right)}\right], & \vee \ge 2\wedge \end{cases} ;$$

$$\mathcal{G}^{\mathcal{F}\,9} = \begin{cases} \frac{\vee^2\sqrt{\wedge}}{2\vee\sqrt{\vee}-(2\vee-\wedge)\sqrt{\wedge}}, & \vee \le s3\cdot\wedge \\ \sqrt{\wedge}\left[\sqrt{\vee} - \sqrt{\vee - \sqrt{\frac{\wedge}{\vee}}\left(2\vee - \wedge\right)}\right], & \vee \ge s3\cdot\wedge \end{cases}$$

$$\mathcal{H}^{\mathcal{F}\,9} = \frac{4\vee^3\wedge}{\left(\vee+\wedge\right)\left(2\vee^2 - \vee\wedge + \wedge^2\right)} ,$$

$$\mathcal{C}^{\mathcal{F}\,9} = \begin{cases} \frac{\vee\left(\vee^2+\wedge^2\right)^2}{(\vee+\wedge)(2\vee^3-2\vee^2\wedge+\vee\wedge^2+\wedge^3)}, & \vee \le s1\cdot\wedge \\ \frac{\sqrt{\vee^2+\wedge^2}}{\vee+\wedge}\left[\sqrt{\vee^2+\wedge^2} - \sqrt{\frac{\vee^3-2\vee^2\wedge+\wedge^3}{\vee}}\right], & \vee \ge s1\cdot\wedge \end{cases} ;$$

$$\mathcal{F}_5^{\mathcal{F}\,9} = \begin{cases} \frac{\vee\left[2\vee^3-3\vee^2\wedge+6\vee\wedge^2-3\wedge^3+\left(2\vee^2-\vee\wedge+\wedge^2\right)\sqrt{(\vee-\wedge)^2+4\wedge^2}\right]}{2(4\vee^3-6\vee^2\wedge+6\vee\wedge^2-\wedge^3)}, & \vee \le s2\cdot\wedge \\ \frac{1}{2}\left[\vee - \wedge + \sqrt{\left(\vee-\wedge\right)^2 + 4\wedge^2} - \sqrt{\frac{2}{\vee}}\cdot\right. \\ \left.\sqrt{\left(\vee^2 - 3\vee\wedge + \wedge^2\right)\sqrt{\left(\vee-\wedge\right)^2 + 4\wedge^2} + \vee^3 - 4\vee^2\wedge + 6\vee\wedge^2 - \wedge^3}\right], \\ \qquad \vee \ge s2\cdot\wedge \end{cases}$$

$$\mathcal{F}_6^{\mathcal{F}\,9} =$$

$$= \begin{cases} \frac{\vee}{2(4\vee^4-4\vee^3\wedge+2\vee^2\wedge^2+2\vee\wedge^3-\wedge^4)}\left[-2\vee^4+8\vee^3\wedge\right. \\ \left.-7\vee^2\wedge^2+4\vee\wedge^3-\wedge^4+\left(2\vee^3-2\vee^2\wedge+3\vee\wedge^2-\wedge^3\right)\sqrt{(\vee-\wedge)^2+4\vee^2}\right], \\ \vee\le t2\cdot\wedge \\ \frac{1}{2}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge-\sqrt{\frac{2}{\vee}}\right. \\ \left.\cdot\sqrt{3\vee^3-2\vee\wedge^2+\wedge^3-(\vee^2+\vee\wedge-\wedge^2)\sqrt{(\vee-\wedge)^2+4\vee^2}}\right],\ \vee\ge t2\cdot\wedge \end{cases}$$

$$\mathcal{F}_7^{\mathcal{F}\,9}=\begin{cases} \frac{\left(\vee^2-\vee\wedge+\wedge^2\right)^2}{\vee(2\vee^2-4\vee\wedge+3\wedge^2)}\ ,\ \vee\le 2\wedge \\ \frac{\sqrt{\vee^2-\vee\wedge+\wedge^2}}{\vee}\left[\sqrt{\vee^2-\vee\wedge+\wedge^2}-\sqrt{\vee^2-3\vee\wedge+2\wedge^2}\right]\ ,\ \vee\ge 2\wedge \end{cases}$$

$$\mathcal{F}_8^{\mathcal{F}\,9}=\begin{cases} \frac{\vee^5}{(2\vee-\wedge)(2\vee^3-4\vee^2\wedge+4\vee\wedge^2-\wedge^3)}\ ,\ \vee\le s3\cdot\wedge \\ \frac{1}{2\vee-\wedge}\left[\vee^2-\sqrt{\vee(\vee-\wedge)(\vee^2-3\vee\wedge+\wedge^2)}\right]\ ,\ \vee\ge s3\cdot\wedge \end{cases}$$

$$\mathcal{F}_{10}^{\mathcal{F}\,9}=\frac{\sqrt{\wedge}}{2}\left[\sqrt{\wedge}+\sqrt{4\vee-3\wedge}-\sqrt{\frac{2}{\vee}}\right.$$
$$\left.\cdot\sqrt{2\vee^2-3\vee\wedge+\wedge^2-(\vee-\wedge)\sqrt{\wedge(4\vee-3\wedge)}}\right]\ .$$

Theorem 6. *Let $t3>1$, $t4>1$ respectively $t5>1$ be the roots of the equations*

$$t^3-t^2-t-1=0,\ t^3-3t^2+2t-1=0,\ t^3-5t^2+4t-1=0\ .$$

The complementary of the Greek means with respect to $\mathcal{F}_{10}$ are:

$$\mathcal{A}^{\mathcal{F}\,10}=$$
$$= \begin{cases} \frac{1}{2(\vee+\wedge)}\left[\vee^2+5\vee\wedge-2\wedge^2-(\vee-\wedge)\sqrt{\wedge(4\vee-3\wedge)}\right]\ ,\ \vee\le 3\wedge \\ \left[\vee+2\wedge+\sqrt{\wedge(4\vee-3\vee)}-\sqrt{\vee^2-8\vee\wedge+9\wedge^2+2(\vee-2\wedge)\sqrt{\wedge(4\vee-3\wedge)}}\right] \\ /4\ ,\ \vee\ge 3\wedge \end{cases}$$

$$\mathcal{G}^{\mathcal{F}\,10}=\frac{1}{2}\sqrt{\frac{\wedge}{\vee}}\left[4\vee-\wedge-\sqrt{\vee\wedge}-\left(\sqrt{\vee}-\sqrt{\wedge}\right)\sqrt{4\vee-3\wedge}\right]\ ;$$

$$\mathcal{H}^{\mathcal{F}\,10}=\frac{1}{4\vee(\vee+\wedge)}\left[2\vee^3+9\vee^2\wedge-2\vee\wedge^2-\wedge^3-\left(\vee^2-\wedge^2\right)\sqrt{\wedge(4\vee-3\wedge)}\right]\ ;$$

$$\mathcal{C}^{\mathcal{F}\,10}=$$
$$\begin{cases} \frac{\vee}{2(\vee+\wedge)(\vee^2+\wedge^2)}\left[2\vee^3+\vee^2\wedge+6\vee\wedge^2-\wedge^3-\left(\vee^2-\wedge^2\right)\sqrt{\wedge(4\vee-3\wedge)}\right]\ , \\ \vee\le t3\cdot\wedge \\ \frac{1}{4(\vee+\wedge)}\left\{2\vee^2+\vee\wedge+3\wedge^2+(\vee+\wedge)\sqrt{\wedge(4\vee-3\wedge)}-\sqrt{2}\right. \\ \cdot\left[4\vee^4-4\vee^3\wedge-3\vee^2\wedge^2+2\vee\wedge^3+5\wedge^4+\left(2\vee^3-\vee^2\wedge-4\vee\wedge^2-\wedge^3\right)\right. \\ \left.\left.\sqrt{\wedge(4\vee-3\wedge)}\right]^{1/2}\right\},\vee\ge t3\cdot\wedge \end{cases}$$

$$\mathcal{F}_5^{\mathcal{F}\,10}=$$

$$= \begin{cases} 2\vee - \wedge + \frac{1}{4\wedge}\left[2\vee + \wedge + \sqrt{\wedge(4\vee - 3\wedge)}\right]\cdot\left[\sqrt{(\vee-\wedge)^2+4\wedge^2} - \vee - \wedge\right], \\ \qquad \vee \le 2\wedge \\ \frac{1}{4}\left[\vee + \sqrt{\wedge(4\vee-3\wedge)} + \sqrt{(\vee-\wedge)^2+4\wedge^2} - \sqrt{2}\right. \\ \left. \cdot\sqrt{\left[\vee+\sqrt{\wedge(4\vee-3\wedge)}\right]\cdot\left[\sqrt{(\vee-\wedge)^2+4\wedge^2}+\vee-4\wedge\right] - \wedge(3\vee-5\wedge)}\right], \\ \qquad \vee \ge 2\wedge \end{cases}$$

$$\mathcal{F}_6^{\mathcal{F}\,10} =$$

$$\begin{cases} \frac{1}{4\vee^2}\left[\wedge\sqrt{\wedge(4\vee-3\wedge)}+\vee^2+2\vee\wedge\right]\cdot\left[\sqrt{(\vee-\wedge)^2+4\vee^2}+\vee-\wedge\right] \\ -\frac{1}{2}\sqrt{\wedge(4\vee-3\wedge)}-\frac{3\vee}{4}+\frac{\wedge^3}{4\vee^2}\,, \ \vee \le t4\cdot\wedge \\ \frac{1}{4}\left\{2\wedge-\vee+\sqrt{\wedge(4\vee-3\wedge)}+\sqrt{(\vee-\wedge)^2+4\vee^2}-\sqrt{2}\right. \\ \cdot\left[\left(2\wedge-\vee+\sqrt{\wedge(4\vee-3\wedge)}\right)\cdot\left(\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee-2\wedge\right)\right. \\ \left.\left.+2\vee^2-9\vee\wedge+9\wedge^2\right]^{1/2}\right\}, \vee \ge t4\cdot\wedge \end{cases} ;$$

$$\mathcal{F}_7^{\mathcal{F}\,10} =$$

$$\begin{cases} \frac{1}{2\vee(\vee^2-\vee\wedge+\wedge^2)}\left[2\vee^4-3\vee^3\wedge+6\vee^2\wedge^2-5\vee\wedge^3\right. \\ \left.+2\wedge^4-\vee(\vee-\wedge)^2\cdot\sqrt{\wedge(4\vee-3\wedge)}\right], \ \vee \le t4\cdot\wedge \\ \frac{1}{4\vee}\left\{2\vee^2-\vee\wedge+2\wedge^2+\vee\sqrt{\wedge(4\vee-3\wedge)}-\sqrt{2}\right. \\ \cdot\left[2\vee^4-2\vee^3\wedge+\vee^2\wedge^2-2\vee\wedge^3+2\wedge^4+\vee\left(2\vee^2-5\vee\wedge+2\wedge^2\right)\right. \\ \left.\left.\cdot\sqrt{\wedge(4\vee-3\wedge)}\right]^{1/2}\right\}, \vee \ge t4\cdot\wedge \end{cases} ;$$

$$\mathcal{F}_8^{\mathcal{F}\,10} = \begin{cases} \frac{1}{2\vee^2(2\vee-\wedge)}\left[2\vee^4+6\vee^3\wedge-11\vee^2\wedge^2+6\vee\wedge^3\right. \\ \left.-\wedge^4-\left(2\vee^3-5\vee^2\wedge+4\vee\wedge^2-\wedge^3\right)\sqrt{\wedge(4\vee-3\wedge)}\right], \ \vee \le t5\cdot\wedge \\ \frac{1}{4(2\vee-\wedge)}\left\{2\vee^2+2\vee\wedge-\wedge^2+(2\vee-\wedge)\sqrt{\wedge(4\vee-3\wedge)}\right. \\ -\sqrt{2}\cdot\left[2\vee^4-20\vee^3\wedge+34\vee^2\wedge^2-18\vee\wedge^3+3\wedge^4\right. \\ \left.\left.+\left(4\vee^3-14\vee^2\wedge+12\vee\wedge^2-3\wedge^3\right)\sqrt{\wedge(4\vee-3\wedge)}\right]^{1/2}\right\}, \ \vee \ge t5\cdot\wedge \end{cases} ;$$

$$\mathcal{F}_9^{\mathcal{F}\,10} = \frac{1}{2\vee(2\vee-\wedge)}\left[2\vee^3+5\vee^2\wedge-7\vee\wedge^2+2\wedge^3-\vee(\vee-\wedge)\sqrt{\wedge(4\vee-3\wedge)}\right] .$$

Theorem 7. *Among the Greek means we have only the following relations:*

$$\mathcal{H}^{\mathcal{A}} = \mathcal{C}\ , \ \mathcal{C}^{\mathcal{A}} = \mathcal{H}\ , \ \mathcal{F}_7^{\mathcal{A}} = \mathcal{F}_9\ , \ \mathcal{F}_9^{\mathcal{A}} = \mathcal{F}_7\ ,$$

$$\mathcal{A}^{\mathcal{G}} = \mathcal{H}\ , \ \mathcal{H}^{\mathcal{G}} = \mathcal{A}\ , \ \mathcal{F}_8^{\mathcal{G}} = \mathcal{F}_9 \ \text{ and } \mathcal{F}_9^{\mathcal{G}} = \mathcal{F}_8\ .$$

Remark 1. As it is shown in [6], each of the above results gives an example of Gaussian double sequence with known limit.

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