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A Characterization of the Golden Section, or of the Constant of Fibonacci

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Abstract. It is well-known that the famous golden number $\alpha = (1+\sqrt{5})/2$ admits the following two representations

$$x(a) = \sqrt{a + \sqrt{a + \sqrt{a + \dots}}}$$
$$y(b) = b + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b + \ddots}}}$$

with $a = b = 1$.

We prove the converse implication, i.e. if a number A admits the both representations $x(a)$ and $y(b)$ with the same parameter $a = b$, then it results obviously that $a = b = 1$ and $A = x(1) = y(1) = \alpha$.

1. Introduction

As well known, the famous golden section $\alpha = (1+\sqrt{5})/2$, closely involved in the analytic structure of Fibonacci numbers F_n and Lucas numbers L_n, admits the following two representations

$$\alpha = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}} \tag{1.1}$$

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}} . \tag{1.2}$$

Of course, the precise signification of these two previous formulas is that if we consider the sequences of real numbers $\{x_n\}$ and $\{y_n\}$ defined by the equalities

$$x_n = \underbrace{\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots + \sqrt{1}}}}}_{n \text{ radicals}} \tag{1.3}$$

$$y_n = 1 + \underbrace{\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots + 1}}}}_{(n-1) \text{ fractions}} , \tag{1.4}$$

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then these are convergent and we have

$$\alpha = \lim_{n\to\infty} x_n,$$

and

$$\alpha = \lim_{n\to\infty} y_n.$$

A more general case is given by the sequences $\{x_n(a)\}$ and $\{y_n(b)\}$ with $a, b > 0$, having the general term described by the equalities:

$$x_n(a) = \underbrace{\sqrt{a + \sqrt{a + \sqrt{a + \ldots + \sqrt{a}}}}}_{n \text{ radicals}} \tag{1.3$'$}$$

$$y_n(b) = b + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b + \ddots + b}}} \quad \Big\}(n-1) \text{ fractions} . \tag{1.4$'$}$$

These sequences also are convergent and analogously define the numbers:

$$x(a) = \sqrt{a + \sqrt{a + \sqrt{a + \ldots}}} \tag{1.1$'$}$$

$$y(b) = b + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b + \ddots}}} \tag{1.2$'$}$$

i.e.

$$x(a) = \lim_{n\to\infty} x_n(a) \qquad \text{and} \qquad y(b) = \lim_{n\to\infty} y_n(b).$$

The formulas (1.1) and (1.2) show us that the constant of Fibonacci α possesses simultaneously two representations: of type (1.1$'$) and of type (1.2$'$), with the same a and b, more, with the "simplest" value of a and b, namely $a = b = 1$.

In this paper we emphasize that the converse affirmation holds: if a real number A possesses simultaneously two representations of the form (1.1$'$) and (1.2$'$) with *the same* parameter a, i.e. $A = x(a) = y(a)$, then it follows obviously that $a = 1$ and $A = \alpha$.

2. A little recall

In order to establish the convergence and the limit of the sequences (1.3$'$) and (1.4$'$) it is necessary to consider and to use the recurrences

$$x_{n+1}(a) = \sqrt{a + x_n(a)} = f(x_n(a)) \qquad (\text{with } x_1(a) = \sqrt{a}) \tag{2.1}$$

and

$$y_{n+1}(b) = b + \frac{1}{y_n(b)} = g(y_n(b)) \qquad (\text{with } y_1(b) = b), \tag{2.2}$$

where the expressions of the functions f and g are obviously $f(x) = \sqrt{a + x}$, for $x > -a$ and $g(y) = b + \dfrac{1}{y}$, for $y > 0$.

So, for the sequence $\{x_n(a)\}$, we obtain the inequalities $x_n(a) < x_{n+1}(a)$ and $0 < x_n(a) < \dfrac{1+\sqrt{1+4a}}{2}$ ([2], A supplement of the Chapter VI), therefore the sequence $\{x_n(a)\}$ is increasing and bounded. So ([1] page 74, or [3], 3.14) the sequence is convergent. Passing to the limit for $n \to \infty$ in (2.1) we find that $x(a)$ is the fixed point of f, i.e. the solution of the equation $x(a) = f(x(a))$, or $x^2(a) - x(a) - a = 0$, for which the unique positive solution is

$$x(a) = \frac{1+\sqrt{1+4a}}{2}. \qquad (2.3)$$

Concerning the sequence $\{y_n(b)\}$, we remark that

$$0 < y_1(b) < y_3(b) < \ldots < y_{2k-1}(b) < \ldots < y_{2k}(b) < \ldots < y_4(b) < y_2(b)$$

i.e. the subsequences $\{y_{2k-1}(b)\}$ and $\{y_{2k}(b)\}$ are both monotone and bounded, therefore convergent. Passing to the limit in the recurrences $y_{2k+1}(b) = (g \circ g)(y_{2k-1}(b))$ and $y_{2k+2} = (g \circ g)(y_k(b))$, we obtain that both the previous sequences have the same limit which is the fixed point of $g \circ g$, i.e. the solution of the equation $y(b) = g(g(y(b)))$. After a little calculation, we obtain that $y(b)$ is the unique positive solution of the equation

$$y^2(b) - by(b) - 1 = 0,$$

namely

$$y(b) = \frac{b+\sqrt{b^2+4}}{2} \qquad (2.4).$$

Of course, for $a = 1$ and $b = 1$, we obtain the convergence of sequences $\{x_n\}$ and $\{y_n\}$ of (1.3) and (1.4), finding the limits (1.1) and (1.2), i.e. $x(1) = y(1) = \alpha$. So, these represent the simplest limits of type (1.1′) and (1.2′), associated to the simpler values of a and b, namely $a = b = 1$.

3. The result

The result which we present now and was shortly announced in our introduction, has a simple aspect and an elementary proof, but we do not meet it in the literature.

Theorem 1. *If $x(a) = y(a)$, then $a = 1$ and $x(a) = y(a) = \alpha$*

Proof. We take into account that the values of $x(a)$ and $y(a)$, given by (2.3) and (2.4) must be equal and we obtain the equation

$$\frac{a+\sqrt{a^2+4}}{2} = \frac{1+\sqrt{1+4a}}{2} \qquad (3.1).$$

This leads us to the solutions $a = 1$ and $a = 0$, which is not acceptable. So $a = 1$ and $x(a) = y(a) = \alpha$. □

The intersection of the graphs of the functions $f(a) = a + \sqrt{a^2+4}$, $a \in \mathbb{R}$ and $g(a) = 1 + \sqrt{1+4a}$, $a \geq -\dfrac{1}{4}$ can illustrate, these solutions more.

So we have obtained that:

The unique real number which admits simultaneously two representations of the form (1.1′) and (1.2′) for the same a is the golden section i.e. the constant of Fibonacci α and $a = 1$.

4. A remark more

The formula (1.2) also can be generalized in another way, namely considering the sequence $\{z_n(n)\}$ defined by the recurrence

$$z_{n+1}(a) = 1 + \frac{a}{z_n(a)} \qquad (\text{with } z_1(a) = 1,\ a > 0). \tag{4.1}$$

So, we obtain

$$z_n(a) = 1 + \left.\frac{a}{1 + \frac{a}{1 + \frac{a}{1 + \frac{a}{1 + \ddots + 1}}}}\right\}(n-1) \text{ fractions}. \tag{4.2}$$

But the right part of (4.2) and also the limit of this sequence, namely

$$z(a) = 1 + \frac{a}{1 + \frac{a}{1 + \frac{a}{1 + \ddots + a}}}$$

do not represent a continued fraction, because the defining value 1 of the "superior layer" of (1.4′) was interchanged with the value b of the "inferior layer" (and are now a). We have so

$$z_{n+1}(a) = 1 + \frac{a}{z_n(a)} \qquad (\text{with } z_1(a) = 1,\ a > 0).$$

The convergence of the sequence $\{z_n(a)\}$ can be established in a similar way as the previous convergence of $\{y_n(b)\}$, but concerning the limit, we obtain (a little surprise!) as for $\{x_n(a)\}$, $z(a) = \lim_{n\to\infty} z_n(a) = \frac{1+\sqrt{1+4a}}{2}$.

So, we have obtained the following

Theorem 2. *For any $a > 0$, the sequences $\{x_n(a)\}$ and $\{z_n(a)\}$ have the same limit*

$$x(a) = z(a) = \frac{1+\sqrt{1+4a}}{2}.$$

(Of course, if $a = 1$, then $x(1) = z(1) = \alpha$).

References

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