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The integer part of some terms of a sequence of real numbers

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Abstract. In this paper we generalize some problems concerning the integer part of some terms of particular sequences of real numbers.

1. Preliminaries

In [2] Chapter 2, paragraph 2.1, are presented the following problems.

3. *Compute the integer part of the number:*

$$A_n = \sqrt[4]{78 + \sqrt[4]{78 + ... + \sqrt[4]{78}}},$$

where there are n radicals.

4. *One consider the numbers:*

$$a_n = \sqrt{6 + \sqrt{6 + ... + \sqrt{6}}};$$

$$b_n = \sqrt[3]{6 + \sqrt[3]{6 + ... + \sqrt[3]{6}}};$$

$$c_n = \sqrt[4]{14 + \sqrt[4]{14 + ... + \sqrt[4]{14}}},$$

every number contains n radicals. Evaluate $[a_n], [b_n], [c_n]$.

5. *Compute* $[A_n]$, *where*

$$A_n = \sqrt{1981 + \sqrt{1981 + ... + \sqrt{1981}}},$$

(A_n contains n radicals).

11.1 *Compute the integer part of the number*

$$a_n = \sqrt{1995 + \sqrt{1995 + ... + \sqrt{1995}}},$$

where there are n radicals.

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2. Main result

These source problems ([1]), suggest the study of a general problem:

Let a and k be given positive integers and let be given the sequence of real numbers $(x_n^{(k)}(a))$ defined by

$$x_n^{(k)}(a) = \sqrt[k]{a + \sqrt[k]{a + ... + \sqrt[k]{a}}},$$

where $x_n^{(k)}(a)$ contains n radicals.

We want to find $[x_n^{(k)}(a)]$, where $[x]$ represents the integer part of the real number x.

We denote by $p \in \mathbb{N}^*$ the integer part of the number $\sqrt[k]{a}$. It results that

$$p \leq \sqrt[k]{a} < p + 1, \ \ p^k \leq a < (p+1)^k$$

and

$$[x_1^{(k)}(a)] = [\sqrt[k]{a}] = p.$$

We consider two cases.

The case 2.1 $p^k \leq a \leq (p+1)^k - p - 1$.
We prove by mathematical induction that

$$p \leq x_n^{(k)}(a) < p + 1 \tag{1}$$

for all positive integers $n \geq 1$.

For $n = 1$ the inequalities (1) are obvious. Now, we consider $n = 2$. We have

$$x_2^{(k)}(a) = \sqrt[k]{a + \sqrt[k]{a}} < \sqrt[k]{(p+1)^k - p - 1 + p + 1} = p + 1$$

and

$$x_2^{(k)}(a) \geq \sqrt[k]{p^k + p} > \sqrt[k]{p^k} = p.$$

Therefore, we can write

$$p < x_2^{(k)}(a) < p + 1.$$

Assuming that the inequalities (1) are true for n, we have

$$x_{n+1}^{(k)}(a) = \sqrt[k]{a + x_n^{(k)}(a)} < \sqrt[k]{(p+1)^k - p - 1 + p + 1} = p + 1$$

and

$$x_{n+1}^{(k)}(a) = \sqrt[k]{a + x_n^{(k)}} \geq \sqrt[k]{p^k + p} > \sqrt[k]{p^k} = p.$$

Hence the inequalities (1) are true for all integer $n \geq 1$.

From (1) it results that

$$[x_n^{(k)}(a)] = p \quad \text{if} \quad \mathrm{p^k \leq a \leq (p+1)^k - p - 1}$$

The case 2.2. $(p+1)^k - p - 1 < a < (p+1)^k$.
We prove now that

$$p+1 \le x_n^{(k)}(a) < p+2 \tag{2}$$

for all positive integer $n, n \ge 2$.

For $n = 2$ we have

$$x_2^{(k)}(a) = \sqrt[k]{a + \sqrt[k]{a}} < \sqrt[k]{(p+1)^k + p + 1} < p+2$$

and

$$x_2^{(k)}(a) = \sqrt[k]{a + \sqrt[k]{a}} \ge \sqrt[k]{(p+1)^k - p + p} = p+1.$$

Assume that the inequalities (2) hold for a certain fixed n and prove that (2) hold for $n+1$.

We have

$$x_{n+1}^{(k)}(a) = \sqrt[k]{a + x_n^{(k)}(a)} < \sqrt[k]{(p+1)^k + p + 2} < p+2$$

and

$$x_{n+1}^{k}(a) = \sqrt[k]{a + x_n^{(k)}(a)} \ge \sqrt[k]{(p+1)^k - p - 1 + p + 1} = p+1,$$

which was to be proved.

From the inequalities (2) we deduce that for $(p+1)^k - p - 1 < a < (p+1)^k$ we have $[x_n^{(k)}(a)] = p+1$ for all positive integers, $n \ge 2$, where $p = [\sqrt[k]{a}]$.

Hence we proved:

Theorem 1. *i) If $p^k \le a \le (p+1)^k - p - 1$, $p = [\sqrt[k]{a}]$, then $[x_n^{(k)}(a)] = p$, for all positive integers k, $k \ge 2$.*

ii) If $(p+1)^k - p - 1 < a < (p+1)^k$, then $[x_1^{(k)}(a)] = p$ and $[x_n^{(k)}(a)] = p+1$, for all positive integers $n, n \ge 2$, and all positive integers $k, k \ge 2$.

3. Particular cases

3.1. $k = 2$ and $p = 1$. The natural numbers a for which $[\sqrt{a}] = p = 1$ are from the interval $[1,4)$, that is $a \in \{1,2,3\}$. Hence $(p+1)^2 - p - 1 = 2$, we deduce that for $a \in \{1,2\}$ we have $[x_n^2(a)] = 1$, for all positive integers n , and for $a = 3$ we have $[x_1^{(2)}(3)] = 1$ and $[x_n^2(3)] = 2$, for all positive integers $n, n \ge 2$. That is we have the identities

$$\left[\sqrt{1 + \sqrt{1 + \sqrt{1 + ... + \sqrt{1}}}}\right] = 1,$$

$$\left[\sqrt{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}\right] = 1,$$

$$\left[\sqrt{3+\sqrt{3+\sqrt{3+...+\sqrt{3}}}}\right]=2,$$

where we have n radicals, $n \geq 2$.

3.2. $k=2$ and $p=2$. The set of the natural numbers for which $[\sqrt{a}]=2$ is $\{4,5,6,7,8\}$. Because $(p+1)^2-p-1=6$ it results that for $a \in \{4,5,6\}$ we have $[x_n^{(2)}(a)]=2$, for all positive integers n, and for $a \in \{7,8\}$ we have $[x_1^{(2)}(a)]=2$ and $[x_n^{(2)}(a)]=3$, for all integer $n \geq 3$. If $a=6$ we have

$$[\sqrt{6+\sqrt{6+...+\sqrt{6}}}]=2,$$

for n radicals, $n \geq 1$, these are the numbers a_n from the problem 4, paragraph 1.

3.3. $a=1981$ and $k=2$. Since $\sqrt{1981}=44,50...$ it follows that $p=[\sqrt{1981}]=44$. Considering that $(p+1)^2-p-1=1980$ and $1981>1980$, from ii) of the Theorem it results that for the sequence

$$x_n^{(2)}(1981)=\sqrt{1981+\sqrt{1981+...+\sqrt{1981}}},$$

we have $[x_1^{(2)}(1981)]=44$ and $[x_n^{(2)}(1981)]=45$, for all positive integers $n, n \geq 2$.

This particular case is the Problem 5 from the paragraph 1.

3.4. $a=1995$ and $k=2$. We obtain $\sqrt{1995}=44,66...$, $p=[\sqrt{1995}]=44$ and $(p+1)^2-p-1=1980$. Because $1995<1980$, it follows that for the sequence

$$x_n^{(2)}(1995)=\sqrt{1995+\sqrt{1995+...+\sqrt{1995}}},$$

we have $[x_1^{(2)}]=44$ and $[x_n^{(2)}(1995)]=45$, for all positive integers $n, n \geq 2$.

This particular case is the problem 11.1 mentioned in the paragraph 1.

3.5. $a=78$ and $k=4$. We have $p=[\sqrt[4]{78}]=2$.

In view of $(p+1)^2-p-1=81-3=78$ and $16<78 \leq 78$ we find by using the case i) from the Theorem that for the sequence

$$x_n^{(4)}(78)=\sqrt[4]{78+\sqrt[4]{78+...+\sqrt[4]{78}}}$$

we have $[x_n^{(4)}(78)]=2$, for all integer $n \geq 1$. This particular case is the problem 3, mentioned in the paragraph 1.

3.6. $a=6$ and $k=3$. Then $p=[\sqrt[3]{6}]=1$ and $(p+1)^3-p-1=6$. Since

$1 < 6 \leq 6$ we are in the case i) from the Theorem and it results that for the sequence

$$x_n^{(3)}(6) = \sqrt[3]{6 + \sqrt[3]{6 + ... + \sqrt[3]{6}}}$$

we have $[x_n^{(3)}(6)] = 1$, for every integer $n \geq 1$.

This particular case is the sequence b_n from the problem 4 mentioned in paragraph 1.

3.7. $a = 14$ and $k = 4$. We obtain $p = [\sqrt[4]{14}] = 1$, $(p+1)^4 - p - 1 = 14$ and $1 < 14 \leq 14$. We are in the case i) from the Theorem and it follows that for the sequence

$$x_n^{(4)}(14) = \sqrt[4]{14 + \sqrt[4]{14 + ... + \sqrt[4]{14}}}$$

we have $[x_n^{(4)}(14)] = 1$, for all integer $n \geq 1$.

This particular case is the sequence c_n from the problem 4, paragraph 1.

3.8. $a = 15$ and $k = 4$. Then $p = [\sqrt[4]{15}] = 2$, $(p+1)^4 - p - 1 = 14$ and $a = 15 > 14$. We have the case ii) from the Theorem and it results that for the sequence

$$x_n^{(4)}(15) = \sqrt[4]{15 + \sqrt[4]{15 + ... + \sqrt[4]{15}}}$$

we obtain $[x_n^{(4)}(15)] = 1$ and $[x_n^{(4)}(15)] = 2$ for any integer $n \geq 2$.

References

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