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About some new means

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Abstract. In this paper we introduce some new means, which give new refinements for Arithmetic Mean-Geometric Mean-Harmonic Mean [AM-GM-HM] inequalities.

1. Introduction

In [2] M. Bencze have introduced a lot of new means which give new refinements for AM-GM-HM etc. inequalities, and presented new methods for generating new means.

In this paper we start with mean $B(x,y) = x - \sqrt{\frac{x^2+y^2}{2}} + y$ introduced by M. Bencze in 1982 (see [1]).

Let use the following notations:

$$A(x_1, x_2, ..., x_n) = \frac{1}{n}\sum_{k=1}^{n} x_k;$$

$$G(x_1, x_2, ..., x_n) = \sqrt[n]{\prod_{k=1}^{n} x_k};$$

$$H(x_1, x_2, ..., x_n) = \frac{n}{\sum\limits_{k=1}^{n} \frac{1}{x_k}};$$

$$Q(x_1, x_2, ..., x_n) = \sqrt{\frac{1}{n}\sum_{k=1}^{n} x_k^2},$$

where $x_k > 0$ $(k = 1, 2, ..., n)$

2. Main Results

Theorem 1. *If* $x, y > 0$, *then the following inequalities hold*

$$A(x,y) \geq B(x,y) \geq G(x,y)$$

Proof. The inequality $A(x,y) \geq B(x,y)$ is equivalent with $\frac{x^2+y^2}{2} \geq \left(\frac{x+y}{2}\right)^2$ or $(x-y)^2 \geq 0$.

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Now we prove

$$B(x,y) \geq G(x,y) \text{ or } x+y \geq \sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy}.$$

Denote $y = t^2 x$, then we have $1+t^2 \geq \sqrt{\frac{1+t^4}{2}} + t$ or $\left(t^2-t+1\right)^2 \geq \frac{1+t^4}{2}$, and finally $(t-1)^4 \geq 0$, which is true.

Equality holds, if and only if $x = y$.

In this case $B(x,y) = 2A(x,y) - Q(x,y)$. Using this remark we introduce the mean

$$B(x_1, x_2, ..., x_n) = \frac{nA(x_1, x_2, ..., x_n) - Q(x_1, x_2, ..., x_n)}{n-1}$$

and obtain the following:

Conjecture 1. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$A(x_1, x_2, ..., x_n) \geq B(x_1, x_2, ..., x_n) \geq G(x_1, x_2, ..., x_n)$$

Proof. (a part). We have proved only the inequality

$$A(x_1, x_2, ..., x_n) \geq B(x_1, x_2, ..., x_n)$$

which is equivalent with

$$Q(x_1, x_2, ..., x_n) \geq A(x_1, x_2, ..., x_n),$$

but this is true.

Now, we introduce the following means:

$$D(x_1, x_2, ..., x_n) = \sqrt[n]{A(x_1, x_2) A(x_2, x_3) ... A(x_n, x_1)}$$

and

$$F(x_1, x_2, ..., x_n) = \sqrt[n]{B(x_1, x_2) B(x_2, x_3) ... B(x_n, x_1)},$$

and prove the following theorem which gives new refinements for AM-GM inequality.

Theorem 2. *If* $x_k > 0$ $(k = 1, 2, ..., n)$, *then*

$$A(x_1, x_2, ..., x_n) \geq D(x_1, x_2, ..., x_n) \geq F(x_1, x_2, ..., x_n) \geq G(x_1, x_2, ..., x_n)$$

Proof. We have

$$A(x_1, x_2, ..., x_n) = \frac{1}{n}\left(\frac{x_1+x_2}{2} + \frac{x_2+x_3}{2} + ... + \frac{x_n+x_1}{2}\right) \geq$$
$$\geq \sqrt[n]{\frac{x_1+x_2}{2} \cdot \frac{x_2+x_3}{2} \cdot ... \cdot \frac{x_n+x_1}{2}} =$$
$$= \sqrt[n]{A(x_1, x_2) A(x_2, x_3) ... A(x_n, x_1)} = D(x_1, x_2, ..., x_n)$$

and

$$\sqrt[n]{G\left(x_1,x_2\right)G(x_2,x_3)...G(x_n,x_1)}=$$

$$=\sqrt[n]{\sqrt{x_1x_2}\cdot\sqrt{x_2x_3}\cdot\ldots\cdot\sqrt{x_nx_1}}=G(x_1,x_2,...,x_n)$$

Using Theorem 1 we obtain the following:

$$A\left(x_1,x_2,...,x_n\right)\geq D\left(x_1,x_2,...,x_n\right)=\sqrt[n]{\prod A\left(x_1,x_2\right)}\geq\sqrt[n]{\prod B\left(x_1,x_2\right)}=$$

$$=F\left(x_1,x_2,...,x_n\right)\geq\sqrt[n]{\prod G\left(x_1,x_2\right)}=G\left(x_1,x_2,...,x_n\right)$$

Denote

$$U_k\left(x_1,x_2,...,x_n\right)=\sqrt[n]{A\left(x_1,x_2,...,x_k\right)A\left(x_2,x_3,...,x_{k+1}\right)...A\left(x_n,x_1,...,x_{k-1}\right)}$$

(so $U_2=D$) and

$$V_k\left(x_1,x_2,...,x_n\right)=\sqrt[n]{B\left(x_1,x_2,...,x_k\right)B\left(x_2,x_3,...,x_{k+1}\right)...B\left(x_n,x_1,...,x_{k-1}\right)}$$

(so $V_2=F$), then we have the following:

Conjecture 2. If $x_i>0\,(i=1,2,...,n)$ and $2\leq k\leq n$, then

$$A\left(x_1,x_2,...,x_n\right)\geq U_k\left(x_1,x_2,...,x_n\right)\geq V_k\left(x_1,x_2,...,x_n\right)\geq G\left(x_1,x_2,...,x_n\right).$$

Proof. (a part). We have

$$A\left(x_1,x_2,...,x_n\right)=\frac{1}{n}\sum\frac{x_1+x_2+...+x_k}{k}\geq\sqrt[n]{\prod\frac{x_1+x_2+...+x_k}{k}}=$$

$$=\sqrt[n]{\prod A\left(x_1,x_2,...,x_k\right)}=U_k\left(x_1,x_2,...,x_n\right)$$

and

$$G\left(x_1,x_2,...,x_n\right)=\sqrt[n]{\prod_{i=1}^{n}x_i}=\sqrt[n]{\prod G\left(x_1,x_2,...,x_k\right)}.$$

From Conjecture 1 we have

$$U_k\left(x_1,x_2,...,x_n\right)=\sqrt[n]{\prod A\left(x_1,x_2,...,x_k\right)}\geq\sqrt[n]{\prod B\left(x_1,x_2,...,x_k\right)}\geq$$

$$\geq\sqrt[n]{\prod G\left(x_1,x_2,...,x_k\right)}=G\left(x_1,x_2,...,x_n\right),$$

so Conjecture 2 is a consequence of Conjecture 1.

Conjecture 3. If $k\geq p$ $(k,p\in\{2,3,...,n\})$ then $U_p\geq U_k\geq V_k\geq V_p$.

Theorem 3. *If* $x,y>0$ *then*

$$G\left(x,y\right)\geq\overline{B}\left(x,y\right)\geq H\left(x,y\right),$$

where

$$\overline{B}\left(x,y\right)=\frac{1}{B\left(\dfrac{1}{x},\dfrac{1}{y}\right)}$$

and this is a new mean which gives a new refinement for GM-HM inequality.

Proof. In Theorem 1, we take $x\to\dfrac{1}{x}$ and $y\to\dfrac{1}{y}$.

Conjecture 4. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$G(x_1, x_2, ..., x_n) \geq \overline{B}(x_1, x_2, ..., x_n) \geq H(x_1, x_2, ..., x_n),$$

where

$$\overline{B}(x_1, x_2, ..., x_n) = \frac{1}{B\left(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}\right)}$$

and represents a new mean which gives a new refinement for GM-HM inequality.

Theorem 5. *If* $x_k > 0\,(k = 1, 2, ..., n)$*, then*

$$G(x_1, x_2, ..., x_n) \geq \overline{F}(x_1, x_2, ..., x_n) \geq \overline{D}(x_1, x_2, ..., x_n) \geq H(x_1, x_2, ..., x_k),$$

where

$$\overline{F}(x_1, x_2, ..., x_n) = \frac{1}{F\left(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}\right)}$$

and

$$\overline{D}(x_1, x_2, ..., x_n) = \frac{1}{D\left(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}\right)},$$

which represent new means and give new refinements for GM-HM inequality.

Proof. We apply Theorem 2 for the numbers $\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}$.

Conjecture 5. If $x_i > 0\,(i = 1, 2, ..., n)$ and $2 \leq k \leq n$, then

$$G(x_1, x_2, ..., x_n) \geq \overline{V_k}(x_1, x_2, ..., x_n) \geq \overline{U_k}(x_1, x_2, ..., x_n) \geq H(x_1, x_2, ..., x_n),$$

where

$$\overline{V_k}(x_1, x_2, ..., x_n) = \frac{1}{V_k\left(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}\right)}$$

and $\overline{U_k}(x_1, x_2, ..., x_n) = \frac{1}{U_k\left(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}\right)}$ represent new means and give new refinements for GM-HM inequality.

References

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