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Modelling quaternions

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Abstract. The Hamilton quaternions are usually introduced as generalisation of complex numbers respecting the basic identities. We present a way to use matrices to introduce quaternions and study their properties, using an isomorphism between the two structures.

1. Preliminaries

Complex numbers are usually introduced in their algebraic form, and the same time connected to the points of the plane identifying $a + bi$ with the point (a, b). If defining the "addition and multiplication of the points of the plane" as follows

$(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac - bd, ad + bc)$ the points of the plane will form a field. Moreover they will induce on the axis OX the usual operations of the real numbers, hence we may consider the operations introduced on the points of the plane as the extension of the similar operations of the real line.

As it is well known, the application $a + bi \mapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is an isomorphism $m : (\mathbb{C}, +, \cdot) \mapsto (\mathbf{M}_2, +, \cdot)$ between the field $(\mathbb{C}, +, \cdot)$ of complex numbers and the field of the 2×2 special form real matrices $(\mathbf{M}_2, +, \cdot)$, where $\mathbf{M}_2 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a, b \in R \right\}$. For simplicity denote the image of $a + bi$ by $m(a + bi)$. We will call $m(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ the matrix model of the complex number $a+bi$, while the field $(\mathbf{M}_2, +, \cdot)$ the model of the field $(\mathbb{C}, +, \cdot)$.

This modelling will be "a true one", operation preserving, and one-to-one mapping between the elements of the two fields.

Indeed we have:

$$f\left((a_1 + b_1 i) + (a_2 + b_2 i)\right) = f((a_1 + a_2) + (b_1 + b_2)i) =$$

$$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ -(b_1 + b_2) & a_1 + a_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix} =$$

$$= f(a_1 + b_1 i) + f(a_2 + b_2 i)$$

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Similarly:

$$f((a_1+b_1i)(a_2+b_2i)) = f(a_1+b_1i)f(a_2+b_2i) =$$

$$= \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix}$$

because

$$(a_1+b_1i)(a_2+b_2i) = (a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)i),$$

$$\begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 - b_1b_2 & a_1b_2 + a_2b_1 \\ -b_1a_2 - a_1b_2 & -b_1b_2 + a_1a_2 \end{bmatrix}$$

and such

$$f\left((a_1+b_1i)(a_2+b_2i)\right) = f(a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)i) =$$

$$= f(a_1+b_1i)f(a_2+b_2i)$$

The same time $f(a_1+b_1i) = f(a_2+b_2i)$ means

$$\begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix},$$

and thus $a_1 = a_2$ and $b_1 = b_2$, in other words $a_1+b_1i = a_2+b_2i$, i.e. the application is an injection, moreover for any matrix $\begin{bmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{bmatrix}$ of M_2 there exists the complex number $a_0 + b_0i$ such as

$$f(a_0+b_0i) = \begin{bmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{bmatrix},$$

i.e. the application is a surjection and hence bijection as well.

We would say that the matrices in M_2 are modelling the complex numbers.

This modelling can be used to study the properties of complex numbers.

First it can be shown that the multiplication of the matrices of the given type is commutative, property which is not generally true for matrices.

Indeed:

$$\begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 - b_1b_2 & a_1b_2 + a_2b_1 \\ -b_1a_2 - a_1b_2 & -b_1b_2 + a_1a_2 \end{bmatrix}$$

and

$$\begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} = \begin{bmatrix} a_2a_1 - b_2b_1 & a_2b_1 + b_2a_1 \\ -b_2a_1 - a_2b_1 & a_2a_1 - b_2b_1 \end{bmatrix}$$

which are the same, as the product of two reals commute.

If we take the subfield of matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, we obtain a model of the real numbers.

The absolute value of a real or a complex number is naturally the square root of the determinant of its matrix model.

If we write

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix},$$

we have nothing else but the trigonometric form of $a + bi = r(\cos t + i \sin t)$, where $r = \sqrt{a^2+b^2}$, and $\cos t = \frac{a}{\sqrt{a^2+b^2}}, \sin t = \frac{b}{\sqrt{a^2+b^2}}$, if considering the right angled triangle with the legs a and b. Here we have a reason for the relation of the complex number with the point (a, b) of the plane, and the same form vector as well.

It is easy to see that the matrix model of conjugate of the complex number $a+bi$ will be the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, we can check on the models properties like:

Proposition 1.1. *Both the sum and the product of a complex number and its conjugate are real.*

Proof.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & 2a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \cdot \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}.$$

The notion of the inverse of a complex number appears now more naturally, as all regular matrices of the given form will have inverses, $det\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a^2+b^2$ which is nonzero, except for $a = 0$ and $b = 0$, i.e. any nonzero complex number has got an inverse.

The inverse of

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ is } \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix},$$

and we are in the position to argue for the well-known

$$(a+bi)^{-1} = \frac{1}{a^2+b^2}(a-bi)$$

Let us try to extend the idea of modelling for extensions of complex numbers. □

2. Generalization

First we take the space coordinate system $OXYZ$.

Is it possible to introduce similar addition and multiplication of the points of the three dimensional space, which will induce on the coordinate planes XOY, XOZ and YOZ the operations of the complex numbers?

It can be proved the contrary, see [1].

Let us suppose that for the point (a, b, c) of the plane there exists an algebraic form $a + bi + cj$, which for $a + bi$ and $a + cj$ will coincide with the operations introduced for complex numbers.

Consider the product of i and j, we will have

$$ij = a_1 + b_1 i + c_1 j$$

Multiplying it from the left by i, the associativity induces:

$$(ii)\,j = i\,(ij) = i\,(a_1 + b_1 i + c_1 j) = a_1 i - b_1 + c_1 ij =$$
$$= a_1 i - b_1 + c_1(a_1 + b_1 i + c_1 j) = (c_1 a_1 - b_1) + (a_1 + c_1 b_1)i + c_1^2 j.$$

But this will mean $i^2 = c_1^2 = -1$, a contradiction for the real number c_1.

As a conclusion it can be stated that for three dimensions there exists no extension of complex numbers.

Remark

In fact the above result is a consequence of the well known Frobenius theorem stating that the real numbers, the complex numbers and the quaternions are the only algebras with the required properties.

It was Hamilton who introduced in 1843 for the points of a four dimensional coordinate system the addition and multiplication, which will induce the usual operations of the complex numbers on the coordinate planes XOY, XOZ, and XOK. This was the first example of noncommutative field, called the field of quaternions.

Let us denote by $1, i, j, k$ a base of the linear space H. The basic operations are:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The product of two elements $a + bi + cj + dk$ and $e + fi + gj + hk$ is introduced as:

$$(a + bi + cj + dk)(e + fi + gj + hk) =$$
$$= (ae - bf - cg - dh) + (af + be + ch - dg)i+$$
$$+(ag + ce - bh + df)j + (ah + de + bg - cf)k.$$

Consequently H is an associative ring with unit $1 + 0i + 0j + 0k$, denoted with 1. From the definition one can deduce that if $a + b + c + d \neq 0$, then the quaternion $x = a + bi + cj + dk$ has got an inverse x^{-1}, where:

$$x^{-1}=\frac{a}{a^2+b^2+c^2+d^2}-\frac{b}{a^2+b^2+c^2+d^2}i-\frac{c}{a^2+b^2+c^2+d^2}j-\frac{d}{a^2+b^2+c^2+d^2}k.$$

Similar to the complex numbers the quaternion $\overline{x} = a - bi - cj - dk$ will be called the conjugate of $x = a + bi + cj + dk$, and it will satisfy similar properties to those of complex conjugates i.e. both the sum and product of a quaternion and its conjugate are real.

In the sequel we will show that the quaternions can be modelled with matrices in a similar way as complex numbers.

Take the bijection

$$a + bi + cj + dk \mapsto \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$$

between $(H, +, \cdot)$, the field of quaternions and the 4×4 real matrices $(M_4, +, \cdot)$ where:

$$M_4 = \left\{ \left[\begin{array}{cccc} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{array} \right] \middle| a, b, c, d \in \mathbb{R} \right\}.$$

This application is injection and surjection, the proof is similar to the previous case.

The same time it will keep the operations as well.

Let us take the quaternions $a + bi + cj + dk$ and $e + fi + gj + hk$, respectively the matrices corresponding them:

$$A = \left[\begin{array}{cccc} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{array} \right] \text{ and } B = \left[\begin{array}{cccc} e & f & g & h \\ -f & e & -h & g \\ -g & h & e & -f \\ -h & -g & f & e \end{array} \right].$$

The product AB of the matrices will be:

$$\left[\begin{array}{cccc} ae - bf - cg - dh & -dg + ch + be + af & ag - bh + ce + df & ah + bg - cf + de \\ -ch + dg - af - be & ae - bf - cg - dh & -bg - ah - de + cf & ag - bh + ce + df \\ -ce - df - ag + bh & ah + bg - cf + de & ae - bf - cg - dh & -ch + dg - af - be \\ -bg - ah - de + cf & -ce - df - ag + bh & -dg + ch + be + af & ae - bf - cg - dh \end{array} \right]$$

The product

$$(a + bi + cj + dk)(e + fi + gj + hk)$$

is equal with

$$(ae - bf - cg - dh) + (af + be + ch - dg)i+$$

$$+(ag + ce - bh + df)j + (ah + de + bg - cf)k.$$

A similar model can be constructed with 2×2 complex matrices, see [2].

The modelling with matrices will have further advantages. The inverse of the quaternion $a + bi + cj + dk$ can be found easily if taking the inverse of the matrix:

$$A = \left[\begin{array}{cccc} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{array} \right]$$

which will be

$$A^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2} \left[\begin{array}{cccc} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{array} \right]$$

in other words the matrix of the quaternion

$$x^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}(a - bi - cj - dk),$$

where

$$x = a + bi + cj + dk.$$

It will be now easy to prove the properties of the inverse.

In spite of the fact that the multiplication of the complex numbers is commutative and the matrix multiplication in M_2 as well, the multiplication of quaternions is not commutative, nor is the product of their matrices.

We can give a counterexample for A and B such as $A \cdot B \neq B \cdot A$ as follows:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$$

In algebraic form it does mean the "strange" result:

$$(1 + i)(j + k) \neq (j + k)(1 + i).$$

Indeed:

$$(1+i)(j+k) = (0-0-0-0)+(0+0+0-0)i+(1+0-1+0)j+(1+0+1-0)k = 0+2k$$

and

$$(j+k)(1+i) = (0-0-0-0)+(0+0+0-0)i+(0+1-0+1)j+(0+1+0-1)k = 0+2j$$

Generally we have only on the main diagonal the same elements in AB and BA:

$$AB - BA = \begin{bmatrix} 0 & -2dg + 2ch & -2bh + 2df & 2bg - 2cf \\ -2ch + 2dg & 0 & -2bg + 2cf & -2bh + 2df \\ -2df + 2bh & 2bg - 2cf & 0 & -2ch + 2dg \\ -2bg + 2cf & -2df + 2bh & -2dg + 2ch & 0 \end{bmatrix}$$

3. Remarks

Other properties can be studied with this matrix model as well to show that they form a skew field isomorphic to the skew field of Hamiltonian quaternions.

One can introduce the conjugated of the quaternion represented by the matrix

$$A = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$$

as the quaternion represented by

$$\overline{A}=\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

The most important properties resulting are $\overline{\overline{A}}=A$, $\left(\overline{A}\right)^{-1}=\overline{A^{-1}}$, more $A+\overline{A}$ and $A\cdot\overline{A}$ are real, that is many of the properties of the complex numbers are inherited. However we will have $\overline{A\cdot B}=\overline{B}\cdot\overline{A}$, and $(A\cdot B)^{-1}=B^{-1}\cdot A^{-1}$ as expected in the noncommutative field.

Take

$$A=\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix},\ \overline{A}=\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

We have

$$A+\overline{A}=\begin{bmatrix} 2a & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix}$$

and

$$A\cdot\overline{A}=\begin{bmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2+c^2+d^2 & 0 & 0 \\ 0 & 0 & a^2+b^2+c^2+d^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2+d^2 \end{bmatrix}$$

If we take two quaternions and their conjugates:

$$A=\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \text{ and } B=\begin{bmatrix} e & f & g & h \\ -f & e & -h & g \\ -g & h & e & -f \\ -h & -g & f & e \end{bmatrix},$$

respectively

$$\overline{A}=\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix},\ \overline{B}=\begin{bmatrix} e & -f & -g & -h \\ f & e & h & -g \\ g & -h & e & f \\ h & g & -f & e \end{bmatrix}$$

we have

$$AB+\overline{B}\cdot\overline{A}=(2ea-2fb-2gc-2hd))\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Final remark.
The modelling of quaternions helps us to understand the fact that during the generalization one will loose some properties, like we have in the case of quaternions, which are not commutative anymore. The idea of further generalization will lead to the structure which is known as Cayley ring. It is also known [1], that in this case one gives up even the associativity, and as a consequence a similar model as the presented one is not suitable to study the alternative structure, the Cayley rings.

References

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