CREATIVE MATH.
14 (2005), 19 - 30

Mathematical induction: Notes for teacher (Part 1)

Michael Lambrou

Abstract. This set of notes is aimed at the teacher of the gifted student who wishes to be exposed to less routine techniques of induction than normally found in a standard school textbook.

The notes that follow are on Mathematical Induction for a classroom of students with a more sophisticated inclination to mathematics than the standard requirement. The article is in two parts. The first includes a historical introduction and the basics of Induction, but the level of mathematics is more or less routine. The second part includes variations of Induction. Here the problems are slightly harder, yet a particular care was taken to include only tractable exercises. Solutions to all problems (except for the standard ones in the first exercise) are given at the end of each part, and are mostly short and elegant.

1. Historical Introduction

In philosophy and in the applied sciences the term *induction* is used to describe the process of drawing general conclusions from particular cases. For Mathematics, on the other hand, such conclusions should only be drawn with caution, because mathematics is a demonstrative science and any statement must be accompanied by a rigorous proof. For example John Wallis (1616-1703) was criticized strongly by his contemporaries because in his *Arithmetica Infinitorum* (1656), after inspecting the six relations,

$$\frac{0+1}{1+1}=\frac{1}{3}+\frac{1}{6} \quad , \quad \frac{0+1+4}{4+4+4}=\frac{1}{3}+\frac{1}{12},$$

$$\frac{0+1+4+9}{9+9+9+9}=\frac{1}{3}+\frac{1}{18} \quad , \quad \frac{0+1+4+9+16}{16+16+16+16+16}=\frac{1}{3}+\frac{1}{24},$$

$$\frac{0+1+4+9+16+25}{25+25+25+25+25+25}=\frac{1}{3}+\frac{1}{30}, \quad \frac{0+1+4+9+16+25+36}{36+36+36+36+36+36+36}=\frac{1}{3}+\frac{1}{36}$$

stated without any further qualification that the general rule, namely,

$$\frac{0^2+1^2+2^2+...+n^2}{n^2+n^2+n^2...+n^2}=\frac{1}{3}+\frac{1}{6n},$$

follows "*per modum inductionis*".

Although Wallis' claim is correct, amounting to the familiar statement (known to Archimedes) that

Received: 07.07.2005. In revised form: 11.07.2005.
2000 *Mathematics Subject Classification.* 97D40, 97D50.
Key words and phrases. *Teaching, induction, problem solving.*

$$1^2 + 2^2 + \ldots + \mathrm{n}^2 = \left(\frac{1}{3} + \frac{1}{6n}\right) n^2(n+1) = \frac{1}{6} n(n+1)(2n+1),$$

it nevertheless needed proof.

One way to deal with this problem is with the so-called method of *complete* or *mathematical induction.* This topic, sometimes called just *induction*, is the subject discussed below.

Induction is a simple yet versatile and powerful procedure for proving statements about integers. It has been used effectively as a demonstrative tool in almost the entire spectrum of mathematics: for example in as diverse fields as algebra, geometry, trigonometry, analysis, combinatorics, graph theory and many others.

The principle of induction has a long history in mathematics. For a start, although the principle itself is not explicitly stated in any ancient Greek text, there are several places that contain precursors of it. Indeed, some historians see the following passage from Plato's (427-347 BC) dialogue *Parmenides* (147a7-c3) as the earliest use of an inductive argument:

Then they must be two, at least, if there is to be contact. - They must. - And if to the two terms a third be added in immediate succession, they will be three, while the contacts [will be] two. - Yes. - And thus, one [term] being continually added, one contact also is added, and it follows that the contacts are one less than the number of terms. For the whole successive number [of terms] exceeds the number of all the contacts as much as the first two exceed the contacts, for being greater in number than the contacts: for afterwards, when an additional term is added, also one contact to the contacts [is added]. - Right. - Then whatever the number of terms, the contacts are always one less. -True.

The previous passage is from a philosophical text. There are, however, several ancient mathematical texts that also contain quasi-inductive arguments. For instance Euclid ($\sim 330 - \sim 265BC$) in his *Elements* employs one to show that every integer is a product of primes.

An argument closer to the modern version of induction is in Pappus' (∼290-∼350 AD) *Collectio.* There the following geometric theorem is proved.

Let AB be a segment and C a point on it. Consider on the same side of AB three semi-circles with diameters AB, AC and CB, respectively. Now construct circles C_n as follows: C_1 touches the three semi-circles; C_{n+1} touches C_n and the semicircles on AB and AC. If d_n denotes the diameter of C_n and h_n the distance of its centre from AB, then $d_n = nh_n$.

The way Pappus proves the theorem is to show geometrically the recurrence relation $h_{n+1}/d_{n+1} = (h_n + d_n)/d_n$. Next, he invokes a result of Archimedes (287 - 212 BC) from his *Book of Lemma's* (Proposition 6) which states that conclusion of the theorem above is true for the case $n = 1$. Coupling this with the recurrence relation, he is able to conclude the case for the general n.

After the decline of Greek mathematics, the Muses flew to the Islamic world. Although induction is not explicitly stated in the works of mathematicians in the Arab world, there are authors who reasoned using a preliminary form of it. For example al - Karaji (953-1029) in his *al-Fakhri* states, among others, the binomial theorem and describes the so called Pascal triangle after observing a pattern from a few initial cases (usually 5). He also knew the formula $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$. About a century later we find similar traces of induction in al-Samawal's ($\sim$1130-$\sim$1180) book *al-Bahir*, where the identity $1^2+2^2+3^2+...+n^2 = n(n+1)(2n+1)/6$ appears.

The first explicit inductive argument in a western source is in the book *Arithmeticorum Libri Duo* (1575) of Francesco Maurolyco (1495–1575). For instance it is shown inductively in this text that the sum of the first n odd integers is equal to the nth square number. In symbols, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, a fact known to the ancient Pythagoreans.

Another early reference to induction is in the *Traité du Triangle Arithmetique* of Blaise Pascal (1623–1662), where the pattern known to-day as 'Pascal's Triangle' is discussed. There the author shows that the binomial coefficients ${}^nC_k$ satisfy ${}^nC_k : {}^nC_{k+1} = (k + 1) : (n - k)$, for all n and k with $0 \leq k < n$. Here the passage from n to $n + 1$ uses ${}^nC_r =^{n-1} C_{r-1} +^{n-1} C_r$.

All the above authors used an intuitive idea about the concept of natural number. This is sufficient for our purposes here, and below we shall follow suit. A characteristic of modern mathematics, however, especially from the late 19th century, was to develop the theory axiomatically. In particular, this was accomplished for the natural numbers by Giuseppe Peano (1858-1932) who published the so called 'Peano's axioms' in 1889, in a pamphlet entitled *Arithmetices principia, nova methodo exposita*. The exact procedure need not concern us here. We only mention that one of the axioms was so designed as to incorporate induction as a method of proof. In other words, the intuitive way to deal with induction below, is actually a legitimate technique. We refer to standard books on abstract algebra for the development of Natural Numbers via the axioms of Peano.

In what follows, the theory is presented in short sections, each with its own exercises. These are rather easy especially at the beginning, but those in the last paragraph are more challenging. *Several questions can be solved by other means, but the idea is to use induction in all of them.*

2. Basics

The principle of mathematical induction is a method of proving statements concerning integers. For example consider the statement $1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$, which we denote by $P(n)$. One can easily verify this for various n, for instance $1^2 = 1 = 1.(1+1)(2.1+1)/6, 1^2+2^2 = 5 = 2.(2+1)(2.2+1)/6, 1^2 + 2^2 + 3^2 = 14 = 3.(3 + 1)(2.3 + 1)/6$ and so forth. Here we verified the statement for the cases $n = 1, n = 2$ and $n = 3$ (in a while we shall see that the last two can be dispensed with) but assume that we have verified it up to the particular value $n = k$. The last statement means that we are certain that for this particular value k we have $1^2 + 2^2 + 3^2 + ... + k^2 = k(k + 1)(2k + 1)/6$. But is the formula true for the case of the next integer $n = k+1$? We claim that it is. To see this, *making use of the fact that we have* $1^2 + 2^2 + 3^2 + ... + k^2 = k(k + 1)(2k + 1)/6$, we argue

$$1^2+2^2+3^2+...+k^2+(k+1)^2 = k(k+1)(2k+1)/6+(k+1)^2 = \quad \text{(by assumption)}$$

$$= (k+1)[k(2k+1) + 6(k+1)]/6$$

$$= (k+1)(k+2)(2k+3)/6,$$

and this last is precisely the original claim for $n = k+1$.

Let us recapitulate: We needed to prove the statement $P(n)$ is true for all integers $n \geq 1$. We first verified it for $n = 1$; then, *assuming* that it is true for $n = k$, we verified it for $n = k+1$. In other words, reiterating our result, the validity of $P(1)$ implies that of $P(2)$; the validity of $P(2)$ implies that of $P(3)$; the validity of $P(3)$ implies that of $P(4)$, *and so on* for all integers $n \geq 1$.

The schema we use in the proof can be summarized symbolically as

$$\begin{array}{l} P(1) \\ P(k) \Rightarrow P(k+1) \\ \hline P(n) \quad true \quad for \quad all \quad n \in \mathbf{N} \end{array}$$

The step $P(k) \Rightarrow P(k+1)$ in the proof is called the *inductive step*; the assumption that $P(k)$ is true, is called the *inductive hypothesis*.

Here is another example.

Example 2.1 (Bernoulli's inequality). Show that if $a > -1$ then $(1+a)^n \geq 1+na$ for all $n \in \mathbf{N}$.

Solution. For $n = 1$ it is a triviality (in fact we get an equality). Assume now validity of the inequality for $n = k$; that is, assume $(1+a)^k \geq 1 + ka$. This is our inductive hypothesis, and we are to show $(1+a)^{k+1} \geq 1 + (k+1)a$. We have

$$(1+a)^{k+1} = (1+a)(1+a)^k \geq (1+a)(1+ka) =$$

$$= 1 + (k+1)a + ka^2 \geq 1 + (k+1)a.$$

This, by the principle of induction, completes the proof. □

As a final remark, the above examples start from $n = 1$. This need not be always the case and there are cases (see exercises) that induction may start at any another integer. The situation is self explanatory and there is no need to qualify it any further.

The next exercises require the verification of a variety of formulae. None of these should present the reader with any difficulty and the exercises are there only to familiarize him/her with the idea of induction. In fact, the reader should try to do several of these exercises mentally.

Exercise 2.1 (routine). Show inductively that each of the following formulae is valid for all positive integers n.

a) $1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2(n+1)^2/4$,

b) $1^4 + 2^4 + 3^4 + \ldots + n^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$,

c) $1^5 + 2^5 + 3^5 + \ldots + n^5 = n^2(n+1)^2(2n^2+2n-1)/12$,

d) $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + ... + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$,

e) $\dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} + ... + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$,

f) $\dfrac{3}{1^2 2^2} + \dfrac{5}{2^2 3^2} + \dfrac{7}{3^2 4^2} + ... + \dfrac{2n+1}{n^2(n+1)^2} = \dfrac{n(n+2)}{(n+1)^2}$,

g) $(n+1)(n+2)...(2n-1)(2n) = 2^n.1.3.5....(2n-1)$,

h) $\sum\limits_{k=1}^{n} \frac{(2k)!}{k!2^k} = \sum\limits_{k=1}^{n} 1 \cdot 3 \cdot 5 \cdot ... \cdot (2k-1)$,

i) $1 - \dfrac{x}{1!} + \dfrac{x(x-1)}{2!} - ... + (-1)^n \dfrac{x(x-1)...(x-n+1)}{n!} = (-1)^n \dfrac{(x-1)(x-2)...(x-n)}{n!}$,

j) $(cosx)(cos2x)(cos4x)(cos8x)...(cos2^{n-1}x) = \dfrac{\sin 2^n x}{2^n \sin x}$

(for $x \in \mathbf{R}$ with $sinx \neq 0$),

k) $\sum\limits_{k=1}^{n} \cos(2k-1)x = \dfrac{\sin 2nx}{2 \sin x}$ $\quad (for\ \ x \in R\ \ with\ \ sinx \neq 0)$,

l) $\underbrace{\sqrt{2 + \sqrt{2 + ... + \sqrt{2 + \sqrt{2}}}}}_{n \text{ radicals}} = 2\cos\dfrac{\pi}{2^{n+1}}$,

m) $(1^5 + 2^5 + 3^5 + \ldots + n^5) + (1^7 + 2^7 + 3^7 + \ldots + n^7) = 2(1+2+3+\ldots+n)^4$,

n) $\dfrac{1}{n+1} + \dfrac{1}{n+2} + ... + \dfrac{1}{2n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + ... + \dfrac{1}{2n-1} - \dfrac{1}{2n}$.

Exercise 2.2. If a sequence (a_n) satisfies

a) $a_{n+1} = 2a_n + 1 (n \in \mathbf{N})$, show that $a_n + 1 = 2^{n-1}(a_1 + 1)$.

b) $a_1 = 0$ and $a_{n+1} = (1-x)a_n + nx (n \in \mathbf{N})$, where $x \neq 0$, show that

$$a_{n+1} = [nx - 1 + (1-x)^n]/x.$$

Exercise 2.3. Let (a_n) be a given sequence. Define new sequences $(x_n), (y_n)$ by $x_1 = 1,\ \ x_2 = a_1,\ y_1 = 0,\ y_2 = 1$ and, for $n \geq 3$, $x_n = a_n x_{n-1} + x_{n-2},\ y_n = a_n y_{n-1} + y_{n-2}$. Show that

$$x_{n+1} y_n - x_n y_{n+1} = (-1)^n.$$

Exercise 2.4. If each of $a_1, a_2, \ldots, a_n$, is a sum of two perfect squares, show that the same is true for their product.

Exercise 2.5. Show that $2n^5/5 + n^4/2 - 2n^3/3 - 7n/30$ is an integer for all $n \in \mathbf{N}$.

Exercise 2.6. Show that if $x \neq y$, then the polynomial $x - y$ divides $x^n - y^n$.

Exercise 2.7. Show that a convex n-gon has

$$1/2n(n-3) \text{ diagonals } \ (n \geq 3).$$

Exercise 2.8. Prove the binomial theorem inductively. Namely, show that

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^k b^{n-k}$$

where ${}^nC_k = \dfrac{n!}{k!(n-k)!}$. You may use ${}^{n+1}C_k =^n C_{k-1} +^n C_k (1 \leq k \leq n)$. (The binomial theorem was known to the Arabs. They did not have a complete proof, but after verifying it for few small n stated the general form using in a quasi-inductive argument. Later the theorem was rediscovered by Isaac Newton (1654-1705), who included it in his celebrated *Philosophiae Naturalis Principia Mathematica* (1687). For the proof he used a combinatorial argument. The first inductive proof was by Jakob Bernoulli(1654-1705), published posthumouslyin his *Ars Conjectandi* (1713)).

Exercise 2.9. It is easy to see that the number $(2+\sqrt{3})^n$ can be written in the form $a_n + b_n\sqrt{3}$. Show a) inductively and b) without induction, that the numbers a_n, b_n satisfy $a_n^2 - 3b_n^2 = 1$ $(n \in \mathbf{N})$.

Exercise 2.10. Show that the number $2^{2^n} - 1$ is divisible by at least n distinct primes.

Exercise 2.11. If $\mathrm{F}_n = a^{2^n} + 1$ is the n^{th} Fermat number $(n = 0, 1, 2, \ldots)$, show that

$$F_n - 2 = (a-1)F_0F_1 \ldots F_{n-1}(n \in \mathbf{N}).$$

Exercise 2.12. Prove by induction that $n! > 3^n$ for $n \geq 7$.

Exercise 2.13. If $a_0, a_1, a_2, \ldots$is a sequence of positive numbers satisfying $a_0 = 1$ and

$$a_{n+1}^2 > a_n a_{n+2}(n = 0, 1, 2, \ldots),$$

show that

$$a_1 > a_2^{1/2} > a_3^{1/3} > a_4^{1/4} > ... > a_n^{1/n} > ....$$

Exercise 2.14. A result of Ramanujan (whose proof is beyond the scope of this article) states that

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+...}}}}} = 3.$$

Use Ramanujan's result to show that for all $n \in \mathbf{N}$,

$$\sqrt{1+n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+...}}}}} = n+1.$$

3. Patterns

One of the disadvantages of the method of induction, as reflected by some of the examples portrayed above (especially in Exercise 1), is that one needs to know *beforehand* the formula describing the situation considered. It is only then that one may embark on proving it. But this need for foreknowledge can often be remedied by detecting patterns after judicial evaluation of special cases. In practice it means that one needs to *conjecture* the underlying rule, and then verify whether it is, indeed, correct. In other words, we have to do some guessing. The following examples elucidate this point.

Example 3.1. For what values on n is $2^n + 1$ a multiple of 3?

Solution. By checking small values of the integer n one realizes that $2^n + 1$ is a multiple of 3 for n equals 1, 3, 5 and 7, but fails to be so when n equals 2, 4, 6 or 8. It seems reasonable to guess that 2^n+1 a multiple of 3 precisely when n is odd. This turns out to be correct, and the following inductive argument can be used (how?) to verify the claim: Write $a_n = 2^n + 1$. Then $a_{n+2} = 2^{n+2} + 1 = 4(2^n + 1) - 3 = 4a_n - 3$, which is a multiple of 3 when a_n is. □

Example 3.2. If $f(x) = 2x + 1$, guess a formula for the nth term of the sequence

$$f_1 = f(x), f_2 = f(f(x)), f_3 = f(f(f(x))), f_4 = f(f(f(f(x)))), \ldots$$

and then prove it by induction.

Solution. By direct calculation one verifies that $f_2 = 4x + 1, f_3 = 8x + 7, f_4 = 16x + 15$ and so on. If these examples are not adequate to guess the pattern, the reader should continue with further iterations of f. Sooner or later one suspects that $f_n = 2^n x + 2^n - 1$. It turns out that the guess is correct, as the reader should supply the missing portions of the following inductive argument that settles the matter:

$$f_{n+1} = f(f_n(x)) = f(2^n x + 2^n - 1) = 2(2^n x + 2^n - 1) + 1 = 2^{n+1} x + 2^{n+1} - 1.$$

□

Example 3.3. By considering the numerical sequence

$$2 - 1, 3 - (2 - 1), 4 - (3 - (2 - 1)), 5 - (4 - (3 - (2 - 1))), \ldots$$

guess and then prove inductively the numerical value of

$$n - (n - 1 - (n - 2 - (n - 3 - (\ldots - (3 - (2 - 1))...)))).$$

Solution. The first few expressions simplify to $1, 2, 2, 3, 3$ and 4 respectively. One may guess that the general pattern is

$$n - (n - 1 - (n - 2 - (n - 3 - (\ldots - (3 - (2 - 1))...)))) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is even} \end{cases}$$

This is easy to verify inductively and the details are left to the reader, who should consider separately the cases n even and n odd. □

A word of caution is necessary here: No matter how many initial cases we check in a particular situation, a pattern that seems to emerge is not sufficient to draw conclusions. A proof must *always* follow our guess and failure to devise such a proof may indicate that our conjecture is, perhaps, wrong. There are several examples showing that even first rate mathematicians were deceived by a few special cases. The great Fermat, for example, after observing that

$$2^{2^0} + 1 = 3, \quad 2^{2^1} + 1 = 5, \quad 2^{2^2} + 1 = 17, 2^{2^3} + 1 = 257$$

and

$$2^{2^4} + 1 = 65537$$

are prime numbers, thought that $2^{2^n} + 1$ is a prime for each n. This turned out to be false, and the first counterexample was given by Euler who found that $2^{2^5} + 1 = 641 \times 6700417$.

Sometimes the first counterexample to what might appear to be a pattern is very far away. For instance, the numbers $n^{17} + 9$ and $(n + 1)^{17} + 9$ are relatively prime for $n = 1, 2, 3, ...$ successively, and for a *very* long time after that. But is this always the case? No, and the first counterexample is for

$$n = 8424432925592889329288197322308900672459420460792433.$$

There are two delightful articles by Richard Guy, entitled *The Strong Law of Small Numbers* (American Mathematical Monthly, (1988) 697-711) and *The Second Strong Law of Small Numbers* (Mathematics Magazine, 63 (1990) 3 - 20) with numerous examples of sequences that *seem* to follow a pattern. But in some cases the reality is, against all intuition, very different. It is worth also looking at the web page

http:// primes.utm.edu/glossary/page.php?sort=LawOfSmall

where the previous example appears.

Here are some exercises along the above lines, where the reader is invited either (i) to discover a pattern and then prove his/her hypothesis correct, or (ii) to find a counterexample that contravenes the pattern that appears at first sight.

Exercise 3.1. After guessing an appropriate formula by testing a few first values of n, use an inductive argument to find the following sums.

a) $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1}n^2$,

b) $1 \cdot (1!) + 2 \cdot (2!) + 3 \cdot (3!) + \ldots + n \cdot (n!)$,

c) $n^2 - [(n-1)^2 - [(n-2)^2 - [(n-3)^2 - [\ldots - [3^2 - (2^2 - 1^2)]\ldots]]]]$,

$$\frac{1}{x(x+1)} + \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \ldots + \frac{1}{(x+n-1)(x+n)}.$$

Exercise 3.2. It is given that the sum $1^6 + 2^6 + 3^6 + \ldots + n^6$ can be simplified in the form

$$n(n+1)(2n+1)(An^4 + Bn^3 - 3n + 1)/42,$$

where A and B are constants independent of n. Guess appropriate values of A and B and then verify that they lead to a valid formula.

Exercise 3.3. If (p_n) is the sequence of primes starting from $p_1 = 2$, show that the sequence of numbers $p_1 + 1,\ p_1p_2 + 1,\ p_1p_2p_3 + 1, \ldots, p_1p_2p_3\ldots p_n + 1$, used by Euclid in a proof in his *Elements*, consists of prime numbers for $n = 1, 2, 3, 4, 5$ but not for $n = 6$.

Exercise 3.4. Given n points on the circumference of a circle, where n is successively $1, 2, 3, 4, \ldots$, draw (in separate figures) all chords joining them. For this make sure that the points are "in general position" in the sense that no three chords are concurrent. Now, count the regions into which each circle is partitioned by the chords. You will find that they are, successively $1, 2, 4, 8, 16, \ldots$ What pattern seems to emerge? Is the next answer 32?. Show that it is not!

Exercise 3.5. Guess the general term of the sequence (a_n) if $a_0 = 1, a_n = 2$ and for $n \geq 1$, $a_{n+1} = \sqrt{a_n + 6\sqrt{a_{n-1}}}$.

Exercise 3.6. Guess the general term of the sequence (a_n) if $a_0 = 1$, and for $n \geq 1$ we have $\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n} = \frac{1}{2}(n+1)\sqrt{a_n}$.

4. Divisibility

The method of induction can be applied to an abundance of situations, not just proving formulae as, perhaps, most of the above examples suggest. In what follows we shall see some of these different circumstances. We start with a fairly easy situation, the case of divisibility of integers, of which we have already seen some exercises in §2. We shall use the notation $a|b$ to signify that an integer a divides, or is a factor of, an integer b.

Example 4.1. Show that for each positive integer n we have $9|5^{2n}+3n-1$; that is, 9 divides the number $5^{2n}+3n-1$.

Solution. Let $a_n = 5^{2n}+3n-1$. It is clear that $a_1 = 27$ is divisible by 9. Assume now that for $n = k$, the number a_n is divisible by 9, that is $5^{2k}+3k-1 = 9M$ for some integer M. We have to show that $a_{k+1} = 5^{2(k+1)}+3(k+1)-1 = 25.5^{2k}+3k+2$ is also divisible by 9. The idea is to somehow use our inductive hypothesis, and this done as follows:

$$a_{k+1} = 25.5^{2k}+3k+2 = 25(5^{2k}+3k-1) - 72n + 27$$

$$= 25.9M - 9(8n-3) \quad \text{(by the inductive hypothesis)}$$

$$= a \text{ multiple of } 9.$$

Therefore by the principle of induction $9|a_n$ for all positive integers n. □

Exercise 4.1. Redo the previous example more elegantly by considering $a_{k+1} - 25a_k$ in place of a_{k+1} alone.

Example 4.2. Show that all numbers in the sequence 1003, 10013, 100113, 1001113,... and so on, are divisible by 17.

Solution. We have $1003 = 17 \times 59$, moreover, the difference between two consecutive numbers of the sequence is of the form $9010\ldots0$, which is also a multiple of 17 ($note 901 = 17 \times 53$). With this information the reader should be able to fill the details of a full inductive argument. □

Exercise 4.2. Show that for each n∈**N**, 7^{2n} – 48n – 1 is a multiple of 2304.

Exercise 4.3. Show that for each n∈**N**, $3.5^{\,2n+1} + 2^{3n+1}$ is a multiple of 17.

Exercise 4.4. Show that the sum of cubes of any three consecutive integers is divisible by 9.

5. Inequalities

We have seen an inequality, Bernoulli's inequality (Example 2.1), that depends on a natural number n. This particular one was proved using induction and, sure enough, many inequalities that depend on n can be dealt with by induction. For instance the following generalization of Bernoulli's inequality can be shown by a minor modification of the proof given above.

Example 5.1 (Weierstrass inequality). If $a_n \geq -1 (n \in \mathbf{N})$, then

$$\prod_{k=1}^{n}(1+a_k) \geq 1 + \sum_{k=1}^{n} a_k$$

Proof. As mentioned, the proof follows closely that of Bernoulli's inequality given above, and the details are left to the reader: For the inductive step then one only needs to multiply both sides by the positive number $(1+a_{n+1})$.

The rest are simple. □

There are several inequalities in the text and in the exercises of what follows, but here is a preliminary set.

Exercise 5.1. Prove by induction that a) $2^n > n^2$ for $n \geq 5$, b) $2^n > n^3$ for $n \geq 10$.

Exercise 5.2. Prove by induction that $2!4!\dots(2n)! > [(n+1)!]^n (n \in \mathbf{N})$.

Exercise 5.3. Prove that $(2n)!(n+1) > 4^n(n!)^2$ for all $n > 1$.

Exercise 5.4. Prove for all integers $n > 1$ the inequality

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$$

Exercise 5.5. Prove that if a_k satisfies $0 < a_k < 1$ for $1 \leq k \leq n$, then

$$(1-a_1)(1-a_2)\dots(1-a_n) > 1 - (a_1 + a_2 + \dots + a_n).$$

Exercise 5.6. Prove that if a_k satisfies $0 \leq a_k \quad \leq 1$ for $1 \leq k \leq n$, then

$$2^{n-1}(1 + a_1a_2\dots a_n) \quad \geq (1+a_1)(1+a_2)\dots(1+a_n).$$

Solutions to the exercises.

2.1) Routine

2.2) The inductive step uses $a_{k+1} + 1 = 2a_k + 2 = 2(a_k + 1) = 2^k(a_1 + 1)$. The second case is just as routine.

2.3) Use $x_{n+1}y_n - x_ny_{n+1} = (a_{n+1}x_n + x_{n-1})y_n - x_n(a_{n+1}y_n + y_{n-1}) = -(x_ny_{n-1} - x_{n-1}y_n)$.

2.4) Use $(x^2 + y^2)(u^2 + v^2) = (xu - yv)^2 + (xv + yu)^2$.

2.5) If $P(k) = 2k^5/5 + k^4/2 - 2k^3/3 - 7k/30$ then, expanding, $P(k+1) = P(k) + integer$.

2.6) Use $x^{n+1} - y^{n+1} = x(x^n - y^n) + y^n(x - y)$.

2.7) It is easy to see that an addition of new vertex to an k-gon increases the number of diagonals by $k-1$ and

$$1/2k(k-3) + k - 1 = (k+1)(k-2).$$

2.8) This is a standard textbook proof.

2.9) a) Use $(2+\sqrt{3})^{n+1} = (a_n + b_n\sqrt{3})(2+\sqrt{3}) = (2a_n + 3b_n) + (a_n + 2b_n)\sqrt{3}$ so that $a_{n+1} = 2a_n + 3b_n$ and $b_{n+1} = a_n + 2b_n$. It is easy now to show that $a_{n+1}^2 - 3b_{n+1}^2 = 1$. b) It is easy to see by the binomial theorem that $(2+\sqrt{3})^n = a_n - b_n\sqrt{3}$. Now use $(2+\sqrt{3})^n(2-\sqrt{3})^n = (4-3)^n = 1$.

2.10) Use $2^{2^{n+1}} - 1 = (2^{2^n} - 1)(2^{2^n} + 1)$.Note that $2^{2^n} - 1$ and $2^{2^n} + 1$ do not have common prime divisors as they are both odd numbers differing by 2.

2.11) The case $n = 1$ is clear. From the hypothesis $F_k - 2 = (a-1)F_0F_1\dots F_{k-1}$ we have

$$F_{k+1} - 2 = a^{2^{k+1}} - 1 = (a^{2^k} - 1)(a^{2^k} + 1) = (F_k - 2)F_k = (a-1)F_0F_1\dots F_{k-1}F_k.$$

2.12) $7! = 5040 > 2187 = 3^7$.If $k! > 3^k(where k \geq 7)$ then $(k+1)! = (k+1)(k!) > (k+1).3^k \quad \geq 8.3^k > 3^{k+1}$.

2.13) The condition $a_1^2 > a_0a_2 = a_2$ gives the first inequality. Assuming $a_{k-1}^{1/(k-1)} > a_k^{1/k}$ we have $a_k^2 > a_{k-1}a_{k+1} > (a_k)^{(k-1)/k}a_{k+1}$, from which the result easily follows.

2.14) The case $n = 1$ is Ramanujan's result.

For the inductive step, let

$$\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+...}}}}} = k+1.$$

Now square both sides, subtract 1 and divide by k. It gives the next case.

3.1) a)$(-1)^n(1+2+...+n) = (-1)^n n(n+1)/2$

b) $(n+1)!$

c) $1+2+...+n = n(n+1)/2$

d) $n/[x(x+n)]$

3.2) $A = 3, B = 6$.

3.3) Initially we find the primes $3, 7, 31, 211, 2311$ but for $n = 6$ the result is $30031 = 59 \times 509$.

3.4) The next number, corresponding to $n = 6$, is 31.

3.5) We find $a_2 = 2^{3/2}$, $a_3 = 2^{7/4}, a_4 = 2^{15/8}$ etc. One may guess and then easily verify by induction that $a_n = 2^{(2^n-1)/2^{n-1}}$.

3.6) It is easy to verify that $a_2 = 4, a_3 = 9$ etc. The guess $a_n = n^2$ is correct and can be verified by induction. A quick way to do this is to verify first that $\sqrt{a_{n+1}} = \dfrac{n+1}{n}\sqrt{a_n}$.

4.1) This essentially the previous example: $a_{k+1} - 25a_k = -9(8k-3)$.

4.2) If $a_n = 7^{2n} - 48n - 1$, for the inductive step consider $a_{k+1} - 49a_k = 2304k$.

4.3) If $a_n = 3.5^{2n+1} + 2^{3n+1}$, the inductive step can be sorted by writing $a_{k+1} - 25a_k = -17 \cdot 2^{2n+1}$

Alternatively, we could consider $a_{k+1} - 8a_k = 3 \cdot 17 \cdot 5^{2k+1}$.

4.4) If $a_k = k^3+(k+1)^3+(k+2)^3$, then $a_{k+1}-a_k = (k+3)^3-k^3 = 9(k^2+3k+3)$.

5.1) a) $2^5 = 32 \geq 5^2$. If $2^k > k^2$ (where $k \geq 5$) then

$$2^{k+1} = 2.2^k > 2.k^2 = k^2 + k^2 \quad \geq k^2 + 5k > k^2 + 2k + 1 = (k+1)^2 .b) 2^{10} =$$

$$= 1024 > 10^3 . If 2^k > k^3 (\text{where} \quad k \geq 10) \ \text{ then } \quad 2^{k+1} =$$

$$2.2^k > 2.k^3 = k^3 + k^3 \quad \geq k^3 + 10k^2 \geq k^3 + 3k^2 + 3k + 1 = (k+1)^3.$$

5.2) The inductive step amounts to showing $(k+2)\ldots(2k+2) > (k+2)^{k+1}$. This is clearly true since each of the $k+1$ terms of the left hand side is $> (k+2)$.

5.3) If $(2k)!(k+1) > 4^k(k!)^2$ then

$$(2k+2)!(k+2) = (2k+2)(2k+1)[(2k)!(k+1)](k+2)/(k+1) >$$

$$> (2k+2)(2k+1)4^k(k!)^2(k+2)/(k+1) =$$

$$4^{k+1}((k+1)!)^2(2k+1)(k+2)/[2(k+1)^2] > 4^{k+1}((k+1)!)^2$$

5.4) The inductive step amounts to showing $\sqrt{n} + \dfrac{1}{\sqrt{n+1}} > \sqrt{n+1}$, which is routine.

5.5) The argument is a trivial adaptation of that of Example 5.1 in the text.

5.6) For the induction step assume validity of the inequality for n = m and any sequence (a_k) with $0 \leq a_k \leq 1$ for $1 \leq k \leq m$. Let now $n = m+1$ and consider a sequence (b_k) with $0 \leq b_k \leq 1$ for $1 \leq k \leq m+1$. Apply the inductive hypothesis to

$a_1 = b_1, a_2 = b_2, \ldots, a_{m-1} = b_{m-1}$ and $a_m = b_m b_{m+1}$.

Thus

$$2^m(1 + b_1 b_2 \ldots b_{m-1}(b_m b_{m+1})) \geq 2(1 + b_1)(1 + b_2)\ldots(1 + b_{m-1})(1 + b_m b_{m+1}).$$

The required result follows upon observing that

$$2(1 + b_m b_{m+1}) \geq (1 + b_m)(1 + b_{m+1})$$

(which is true as equivalent to the true statement $(1 - b_m)(1 - b_{m+1}) \geq 0.$)

Apart from the books mentioned in the text, we recommend in References some articles for further historical aspects of Induction.

References

[1] Rabinovitch N. L, Rabbi Levi Ben Gershon and the Origins of Mathematical Induction, Archive for History of Exact Sciences, 6 (1970), 237-248

[2] Rashed R., *L' induction mathématique: al_Karaji, as-Samaw'al,* Archive for the History of Exact Sciences, 9 (1972), 1-21

University of Crete
71409 Iraklion
Greece
E-mail address: lambrou@math.uoc.gr