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On the limits of some sequences of integrals

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Abstract. The aim of this paper is to give two general results regarding the computation of some particular limits of real sequences which are defined by a definite integral (Proposition 1 and Proposition 2). Several applications are then presented.

In this article we will present a general method for calculating limits of the form:

$$\lim_{n\to\infty} n \int_0^1 x^n g\left(x^n\right) f\left(x\right) dx.$$

Proposition 1. *If* $f, g : [0,1] \to \mathbb{R}$ *are continuous functions, then:*

$$\lim_{n\to\infty} \int_0^1 g\left(x^n\right) f\left(x\right) dx = g\left(0\right) \int_0^1 f\left(x\right) dx.$$

Proof. If at least one of the functions f or g is identically null, the conclusion is obvious. Otherwise, let denote

$$M_1 = \max\{\left|f(x)\right| / x \in [0,1]\},$$
$$M_2 = \max\{\left|g(x)\right| / x \in [0,1]\}.$$

Then $M_1 > 0$ and $M_2 > 0$. Let $\varepsilon > 0$, given. We consider

$$\alpha \in \left(0,1\right), \alpha < \frac{\varepsilon}{4M_1M_2}$$

.

Since g is continuous at $x = 0$, for any $\varepsilon > 0$, $(\exists)\,\delta\left(\varepsilon\right) > 0$ such that

$$\left|g\left(x\right) - g\left(0\right)\right| < \frac{\varepsilon}{2M_1}.$$

For any $x \in [0, 1-\alpha]$, we have $\lim\limits_{n\to\infty} x^n = 0$. So, there exists $n_\varepsilon \in \mathbb{N}$ for which

$$0 \le x^n < \delta\left(\varepsilon\right), (\forall)\, x \in [0,\, 1-\alpha]\,,\ (\forall)\ n \ge n_\varepsilon.$$

Then:

$$\int_0^{1-\alpha} \left|g\left(x^n\right) - g\left(0\right)\right| \cdot \left|f\left(x\right)\right| dx \le \frac{\varepsilon}{2M_1} \int_0^{1-\alpha} \left|f\left(x\right)\right| dx \le$$

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$$\leq \frac{\varepsilon}{2M_1}\int\limits_0^1 |f(x)|\;dx \leq \frac{\varepsilon}{2M_1}\cdot M_1 = \frac{\varepsilon}{2}.$$

Using the fact that, on the other hand,

$$|g(x^n)-g(0)| \leq |g(x^n)|+|g(0)| \leq 2M_2,$$

$$\int\limits_{1-\alpha}^1 |g(x^n)-g(0)|\cdot|f(x)|\,dx \leq 2M_2\int\limits_{1-\alpha}^1 |f(x)|\,dx \leq$$

$$\leq 2M_2M_1\int\limits_{1-\alpha}^1 1dx = 2M_1M_2\cdot\alpha < \frac{\varepsilon}{2} \tag{2}$$

Now, using (1) and (2), we get:

$$\left|\int\limits_o^1 g(x^n)\,f(x)\,dx - g(0)\int\limits_0^1 f(x)\,dx\right| =$$

$$= \left|\int\limits_0^1 (g(x^n)-g(0))\cdot f(x)\,dx\right| \leq \int\limits_0^1 |g(x^n)-g(0)|\cdot|f(x)|\,dx =$$

$$= \int\limits_0^{1-\alpha} |g(x^n)-g(0)|\cdot|f(x)|\,dx + \int\limits_{1-\alpha}^1 |g(x^n)-g(0)|\cdot|f(x)|\,dx < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This shows that

$$\lim_{n\to\infty}\int\limits_0^1 g(x^n)\,f(x)dx = g(0)\int\limits_0^1 f(x)\,dx.$$

□

Proposition 2. *If* $f, g : [0,1] \to \mathbb{R}$ *are two functions such that* g *is continuous on* $[0,1]$ *and* f *is differentiable, with* f' *continuous on* $[0,1]$*, then:*

$$\lim_{n\to\infty} n\int\limits_0^1 x^n g(x^n)\,f(x)\,dx = f(1)\int\limits_0^1 g(x)\,dx.$$

Proof. Since g continuous, we consider G:$[0,1] \to \mathbb{R}$ a primitive of g. Integrating by parts, we have

$$n\int\limits_0^1 x^n g(x^n)\,f(x)\,dx = \int\limits_0^1 (G(x^n))'\cdot(xf(x))\,dx =$$

$$= G\left(x^n\right) \cdot xf\left(x\right)\big|_0^1 - \int_0^1 G\left(x^n\right)\left(xf\left(x\right)\right)' dx =$$

$$= G(1)f(1) - \int_0^1 G\left(x^n\right) \cdot \left(xf\left(x\right)\right)' dx. \tag{3}$$

By letting $n \to \infty$ and applying Proposition 1, for $g := G$ and $f := (xf)'$, we obtain:

$$\lim_{n\to\infty} n \int_0^1 x^n g\left(x^n\right) f\left(x\right)\, dx =$$

$$= G\left(1\right) f\left(1\right) - G\left(0\right) \int_0^1 \left(xf\left(x\right)\right)' dx =$$

$$= G\left(1\right) f\left(1\right) - G\left(0\right) f\left(1\right) = f(1)[G(1) - G(0)] = f\left(1\right) \int_0^1 g\left(x\right) dx\ .$$

□

Using Proposition 1 or Proposition 2 we can now solve several related problems.

Problem 1
If $f : [0, 1] \to \mathbb{R}$ is continuous, then:

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0 \qquad [1]$$

Solution. Apply Proposition 1 with $g(x) = x$.

Problem 2
If $f : [0, 1] \to \mathbb{R}$ is differentiable and its first derivate is continuous on $[0, 1]$, then:

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = f(1). \qquad [4]$$

Solution. We consider $g(x) = 1$ in Proposition 2.

Problem 3
If $g : [0, 1] \to \mathrm{R} is a continuous function, then : \lim_{n\to\infty} \int_0^1 x^n g\left(x^n\right)\, dx = \int_0^1 g\left(x\right)\, dx$

(E. Păltănea, [3])

Solution. We consider $f(x) = 1$ in Proposition 2.

Now we present some other applications.

1) If $g : [0,1] \to R$ is a continuous function, then:

$$\lim_{n\to\infty} \int_0^1 \frac{g(x^n)}{x+1} dx = g(0) \ln 2.$$

(A. Corduneanu, [2])

Solution.

We consider $f(x) = \dfrac{1}{x+1}$ and apply Proposition 1.

2) If $a > 0$, show that

$$\lim_{n\to\infty} n \int_0^1 \frac{x^n}{a+x^n} dx = \ln \frac{a+1}{a},$$

where $a > 0$.

(O.J.-National Olimpiad, County round, 2001, partial statement)

Solution .

Consider the functions

$$g : [0,1] \to R, \quad g(x) = \frac{1}{a+x},$$

and

$$f : [0,1] \to R, f(x) = 1$$

and apply Proposition 2.

3) Show that

$$\lim_{n\to\infty} n \int_0^1 \frac{x^{2n}}{1+x} dx = \frac{1}{4}.$$

(C. Moanţă, G.M. 12/1999)

Solution.

We apply Proposition 2 for $f(x) = \dfrac{1}{x+1}$ and $g(x) = x$.

4) Evaluate

$$\lim_{n\to\infty} n \int_0^1 \frac{2x^{2n}+x^n}{x^{2n}+x^n+a} dx, \ a > 0.$$

(Florin Rotaru, G.M. 7/2004)

Solution

We consider

$$g : [0,1] \to R, \ g(x) = \frac{2x+1}{x^2+x+a}$$

and

$$f : [0,1] \to R, f(x) = 1.$$

Under the assumption of Proposition 2, we obtain the required limit as:

$$ln\frac{a+2}{a}.$$

5) Compute

$$\lim_{n\to\infty} n\int\limits_0^{\frac{\pi}{4}} \frac{tg^n x + tg^{2n} x}{(1 + tg^n x + tg^{2n} x)\cos^2 x} dx.$$

(Florin Rotaru, G.M.2/2004)

Solution.

Denote $tgx = t$. Then

$$I_n = \int_0^1 \frac{t^n(1+t^n)}{1+t^n+t^{2n}} dt$$

For $g(t) = \dfrac{1+t}{1+t+t^2}$ and $f(t) = 1$, by Proposition 2, we get:

$$\lim_{n\to\infty} I_n = \int\limits_0^1 g(t)\, dt = \frac{\ln 3}{2} + \frac{\pi}{6\sqrt{3}}.$$

6) Compute

$$\lim_{n\to\infty} n\int\limits_0^1 \frac{\left(2x^{2n} + x^n\right) a^x}{x^{2n} + x^n + a} dx, \text{ where } a > 0, a \neq 1.$$

Solution.

Applying Proposition 2 for $g(x) = \dfrac{2x+1}{x^2+a}$ and $f(x) = a^x$, we obtain that the required limit is: a $\ln\dfrac{a+2}{a}$.

We invite the readers to find other interesting applications of Proposition 1 and Proposition 2.

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