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About some mean-value theorems

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Abstract. Is this note we will demonstrate a mean-value theorem (Theorem 3), from which through several particularizations, we will obtain the mean-value theorem of Dimitrie Pompeiu and the mean-value theorem of Mircea Ivan. We will also give a geometrical interpretation of Theorem 3.

1. Introduction

We remind two mean-value theorems, that will be then generalized.

In [4] Dimitrie Pompeiu states the following mean-value theorem

Theorem 1. (Dimitrie Pompeiu, 1946). *Let $f : [a,b] \to \mathbb{R}$ be a function that satisfies the following conditions:*

a) it is continuous on $[a,b]$;
b) it is derivable on (a,b);
c) $0 \notin [a,b]$.
Then exists $c \in (a,b)$ so that

$$\frac{af(b) - bf(a)}{a-b} = f(c) - cf'(c). \tag{1}$$

Another theorem is announced and demonstrated in [2].

Theorem 2. (Mircea Ivan, 1970). *Let $f : [a,b] \to \mathbb{R}$ be a function that satisfies the following conditions:*

a) it is continuous on $[a,b]$;
b) it is derivable on (a,b);
c) $f'(x) \neq 0$, $\forall\, x \in (a,b)$;
d) $f(a) \neq f(b)$.
Then exists $c \in (a,b)$ so that

$$\frac{af(b) - bf(a)}{f(b)-f(a)} = c - \frac{f(c)}{f'(c)}\,. \tag{2}$$

2. Main result

In the following, we consider $a < b$.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a function that satisfies the following conditions:*

(i) it is continuous on $[a,b]$;
(ii) it is derivable on (a,b);

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(iii) $f'(x) \neq 0$, $\forall\, x \in (a,b)$
and consider the points $A(a, f(a))$ *and* $B(b, f(b))$.

If the point $M(\alpha,\beta)$ *satisfies the conditions* $M \in AB$ *and* $M \notin [AB]$, *then there exists* $c \in (a,b)$ *so that:*

$$\alpha\left(f'(c) - \frac{f(a)-f(b)}{a-b}\right) = \frac{af(b)-bf(a)}{a-b} - (f(c) - cf'(c)) \tag{3}$$

and

$$\beta\left(\frac{1}{f'(c)} - \frac{1}{\dfrac{f(a)-f(b)}{a-b}}\right) = \frac{af(b)-bf(a)}{f(b)-f(a)} - \left(c - \frac{f(c)}{f'(c)}\right). \tag{4}$$

Proof. We prove that $f(a) \neq f(b)$. Supposing that $f(a) = f(b)$, then according to Rolle's theorem there exists $c \in (a,b)$ so that $f'(c) = 0$, which is in contradiction with (iii).

We consider the functions

$$g,\, h : [a,b] \to \mathbb{R},\; g(x) = \frac{f(x)}{x-\alpha},\; h(x) = \frac{1}{x-\alpha}.$$

We apply Cauchy's theorem to g and h functions, so exists $c \in (a,b)$ so that

$$\frac{g(b)-g(a)}{h(b)-h(a)} = \frac{g'(c)}{h'(c)},$$

equivalent with

$$\frac{\dfrac{f(b)}{b-\alpha} - \dfrac{f(a)}{a-\alpha}}{\dfrac{1}{b-\alpha} - \dfrac{1}{a-\alpha}} = \frac{\dfrac{f'(c)(c-\alpha) - f(c)}{(c-\alpha)^2}}{\dfrac{1}{(c-\alpha)^2}},$$

from which it results relation (3).

Knowing that the equation for the line AB is

$$y = \frac{f(a)-f(b)}{a-b}x + + \frac{af(b)-bf(a)}{a-b}$$

and that $M \in AB$, we obtain:

$$\alpha = \frac{\beta(a-b) + bf(a) - af(b)}{f(a)-f(b)}. \tag{5}$$

Replacing α from (5) in (3), and after calculating we have (4). □

Remark 1. Under the assumptions of Theorem 3, for $\alpha = 0$, we obtain relation (1) from Theorem 1.

Remark 2. Under the assumptions of Theorem 3, for $\beta = 0$, we obtain relation (2) from Theorem 2.

In the following we will give *the geometrical interpretation of Theorem 3*

This theorem states that there exists a point $T\,(c, f(c))$, $c \in (a, b)$, so that the tangent line in T to the graphic of the function f crosses the line AB in the point M.

Indeed, considering the set of equations formed by the line AB and the tangent line in T to the graphic of the function f,

$$\begin{cases} y = \dfrac{f(a) - f(b)}{a - b} x + \dfrac{af(b) - bf(a)}{a - b} \\ y = f'(c) \cdot x + f(c) - cf'(c) \end{cases} \tag{6}$$

and solving it, we obtain that the solution is $x = \alpha$ and $y = \beta$ given by (3) and (4), meaning that the point $M(\alpha, \beta)$ is the intersection of the two lines.

Remark 3. For $\alpha = 0$ and $\beta = 0$, we obtain the geometrical interpretation of Theorem 1 and Theorem 2.

Remark 4. The result from Theorem 3, relation (3), is contained in Theorem 1 from [6] and is generalized in Theorem 3 from [5].

Remark 5. The result from Theorem 3, relation (4), is contained in Theorem 5 from [5].

Remark 6. Theorem 3 from this paper unifies Theorem 3 from [5] and Theorem 1 from [6].

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