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About Bernstein polynomial and the Stirling's numbers of second type

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Abstract. In this paper we want to determine the coefficients of Bernstein polynomial associated to the functions $e_k(x) = x^k$, $k \in \mathbb{N}$ as well as the degree of this polynomial.

1. Preliminaries

Let $B_m : C[0,1] \to C[0,1]$, $m \in \mathbb{N}^*$ be the Bernstein operators, defined for any function $f \in C[0,1]$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials, defined by

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k},\ k \in \{0,1,\ldots,m\} \text{ and } x \in [0,1]. \tag{1.2}$$

For $x \in \mathbb{R}$, $k \in \mathbb{N}$, let $x^{[k]} = x(x-1)\ldots(x-k+1)$, $x^{[0]} = 1$.

It is well known that

$$x^k = \sum_{\nu=1}^{k} S(k,\nu) x^{[\nu]},\ x \in \mathbb{R},\ k \in \mathbb{N}^*, \tag{1.3}$$

where $S(k,\nu)$, $\nu \in \{1,2,\ldots,k\}$ are the Stirling's numbers of second type.
These numbers verify the relations

$$\begin{aligned} &S(p,k) = kS(p-1,k) + S(p-1,k-1),\ S(1,1) = 1, \\ &S(2,1) = S(2,2) = 1,\ S(p,1) = S(p,p) = 1 \end{aligned} \tag{1.4}$$

for $p \in \mathbb{N}$, $p \geq 3$, $k \in \{2,3,\ldots,p-1\}$, and then can be calculated with (1.4)

					1					
				1		1				
			1		3		1			
		1		7		6		1		
	1		15		25		10		1	
1		31		90		65		15		1

...

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We denote $S(p,k)=0$ from definition if $p,k\in\mathbb{N}$ with $p<k$.

2. Main results

Proposition 2.1. *If* $m,p\in\mathbb{N}^*$, *then*

$$(B_m e_p)(x)=\frac{1}{m^p}\sum_{k=1}^{p} m^{[k]}S(p,k)x^k. \tag{2.1}$$

Proof. We have

$$\begin{aligned}(B_m e_p)(x)&=\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}\frac{k^p}{m^p}=\\&=\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}\frac{1}{m^p}\sum_{\nu=1}^{p}S(p,\nu)k^{[\nu]}=\\&=\frac{1}{m^p}\sum_{\nu=1}^{p}S(p,\nu)x^{\nu}\sum_{k=0}^{m}\binom{m}{k}x^{k-\nu}(1-x)^{m-k}k^{[\nu]}=\\&=\frac{1}{m^p}\sum_{\nu=1}^{p}S(p,\nu)m^{[\nu]}x^{\nu}\sum_{k=\nu}^{m}\binom{m-\nu}{k-\nu}x^{k-\nu}(1-x)^{m-k}=\\&=\frac{1}{m^p}\sum_{\nu=1}^{p}S(p,\nu)m^{[\nu]}x^{\nu}(x+1-x)^{m-\nu}=\\&=\frac{1}{m^p}\sum_{\nu=1}^{p}S(p,\nu)m^{[\nu]}x^{\nu}\end{aligned}$$

so (2.1) holds for $m\ge p$. If $p>m$ (2.1) holds, too, because $k^{[\nu]}=0$, $k,\nu\in\mathbb{N}^*$, $k<\nu$. □

3. Applications

Application 3.1. $(B_m e_1)(x)=\dfrac{1}{m}m^{[1]}S(1,1)x=x$, $m\ge 1$.

Application 3.2. $(B_m e_2)(x)=\dfrac{1}{m^2}\left(m^{[1]}S(2,1)x+m^{[2]}S(2,2)x^2\right)=$

$$=\frac{m-1}{m}x^2+\frac{1}{m}x,\ m\ge 2.$$

Application 3.3.

$$(B_m e_3)(x)=\frac{1}{m^3}\sum_{k=1}^{3}m^{[k]}S(3,k)x^k=$$

$$=\frac{(m-1)(m-2)}{m^2}x^3+\frac{3(m-1)}{m^2}x^2+\frac{1}{m^2}x, m\ge 3.$$

Application 3.4. $(B_3 e_4)(x)=\dfrac{1}{3^4}\displaystyle\sum_{k=1}^{4}S(4,k)3^{[k]}x^k=\dfrac{12}{27}x^3+\dfrac{14}{27}x^2+\dfrac{1}{27}x.$

Corollary 3.1. *The degree of* $(B_m e_p)(x)$ *is* p *if* $m \geq p$ *and* m *if* $m < p$.

Proof. We have $m^{[k]} = 0$ for $k > m$ and all results from (2.1). □

In the paper [2] is proved the following

$$(B_m e_n)(x) = \frac{1}{m^{n-1}}\, x + \frac{(m-1)!}{m^{n-1}} \sum_{k=1}^{n-1} \frac{x^{k+1}}{(m-k-1)!} \cdot \tag{3.1}$$

$$\cdot \sum_{i_1=k}^{n-1} \sum_{i_2=k-1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \binom{n-1}{i_1}\binom{i_1-1}{i_2} \cdots \binom{i_{k-1}-1}{i_k},$$

for any natural numbers m, n and any $x \in [0,1]$.

Corollary 3.2. *We have*

$$S(p, k+1) = \sum_{i_1=k}^{p-1} \sum_{i_2=k-1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \binom{p-1}{i_1}\binom{i_1-1}{i_2} \cdots \binom{i_{k-1}-1}{i_k}, \tag{3.2}$$

where $k \in \{1, 2, \ldots, p-1\}$ *and* $p \in \mathbb{N}$, $p \geq 3$.

Proof. The coefficient of x^{k+1} from (2.1) is $\dfrac{m^{[k+1]}}{m^p} S(m, k+1)$ and from (3.2) is

$$\frac{(m-1)!}{m^{p-1}} \, \frac{1}{(m-k-1)!} \sum_{i_1=k}^{p-1} \sum_{i_2=k-1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \binom{p-1}{i_1}\binom{i_1-1}{i_2} \cdots \binom{i_{k-1}-1}{i_k} =$$

$$= \frac{m^{k+1}}{m^p} \sum_{i_1=k}^{p-1} \sum_{i_2=k-1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \binom{p-1}{i_1}\binom{i_1-1}{i_2} \cdots \binom{i_{k-1}-1}{i_k}$$

so that (3.2) holds. □

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