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A method to determine all non-isomorphic groups of order 12

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Abstract. The present work gives a method to determine of all non-isomorphic groups of order 12 and gives descriptions of all these groups.

In [6] Purdea has determined all non-isomorphic groups of order $n \leq 10$, and I have determined in [9] all these groups of order $n \in \{p, q, pq, p^2, p^3\}$, where p and q are two distinct prime numbers.

In this work we will present a method to determine all non-isomorphic groups of order 12. In this context, throughout this paper by group we mean a group (denoted by G) of order 12 in multiplicative notation and we will denote: by 1 the identity (the "neutral") element of G, by $\mathrm{ord}(g)$ the order of the element $g \in G$ and by $|A|$ the cardinal of the set A. If A is a subgroup of G then $|A|$ is (also) the order of A.

Following the same reasoning as in [9] we will prove the main result of this paper:

Theorem: *There are* 5 *non-isomorphic groups of order* 12.

Proof: So, let G be a group of order 12 and $Z(G)$ his center. According to Lagrange's theorem we have the following cases:

Case I: $|Z(G)| = 12$. In this case G is commutative and by [3,8.4] and [5,2.2 (p. 86) and 6.1 (p. 97)] it follows that either $G = G_1 \cong \mathbf{Z}_{12} \cong \mathbf{Z}_4 \times \mathbf{Z}_3$ or $G = G_2 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_3$. So, either there is an element $x \in G_1$ such that $\mathrm{ord}(x) = 12$ and $G_1 = \{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}\}$-for this group see Table 1, or there are $x, y, z \in G_2$ such that $\mathrm{ord}(x) = \mathrm{ord}(y) = 2$, $\mathrm{ord}(z) = 3$, $xy = yx$, $xz = zx$, $yz = zy$ and $G_2 = \{1, x, y, z, z^2, xy, xz, xz^2, yz, yz^2, xyz, xyz^2\}$-for this group see Table 2.

Case II: $|Z(G)| = 6$. Then $|G/Z(G)| = 2$ and since the group $G/Z(G)$ is cyclic, by [5,2.2 (p. 143)] it follows that G is commutative-contradiction to the hypothesis. So, there is no group G of order 12 with $|Z(G)| = 6$.

Case III: $|Z(G)| = 4$. Also in this case the group $G/Z(G)$ is cyclic and using again [5,2.2 (p. 143)] it follows that G is commutative-contradiction to the hypothesis. Therefore there is no group G of order 12 with $|Z(G)| = 4$.

Case IV: $|Z(G)| = 3$. In this case $|G/Z(G)| = 4$ and according to [8,5.5] the group $G/Z(G)$ is commutative. Again by [3,8.4] and [5,2.2 (p. 86) and 6.1 (p. 97)] it follows that either $G/Z(G) \cong \mathbf{Z}_4$ or $G/Z(G) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. Since the group $\mathbf{Z}_4$ is cyclic, it follows that only the second possibility holds.

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Therefore $G/Z(G) = \{Z(G), xZ(G), yZ(G), xyZ(G)\}$ and $x^2 \in Z(G)$, $y^2 \in Z(G)$, $xyZ(G) = yxZ(G)$. Since $Z(G)$ is a cyclic subgroup of order 3 of G assume $Z(G) = \{1, a, a^2\}$, with $a \in G$. Hence $G = \langle x, y, a\rangle$. If $xy = yx$ then G is commutative, which is impossible. So, $xy \neq yx$ and there is an element $b \in \{a, a^2\}$ such that $yx = xyb$ and $xy = yxb^2$. Now we distinguish the following subcases:

1) $x^2 = y^2 = 1$. Then $(yx)^2 = yxyx = y(xy)x = y(yxb^2)x = y^2x^2b^2 = b^2$ and $(xy)^2 = xyxy = x(yx)y = x(xyb)y = x^2y^2b = b$. On the other hand, $(yx)^2 = yxyx = (xyb)(xyb) = (xy)^2b^2 = b \cdot b^2 = 1$. It follows that $b = 1$-contradiction to the hypothesis.

2) $x^2 = 1$ and $y^2 = c \in \{a, a^2\}$. Then $(yx)^2 = yxyx = y(xy)x = y(yxb^2)x = y^2x^2b^2 = b^2c$ and $(xy)^2 = xyxy = x(yx)y = x(xyb)y = x^2y^2b = bc$. On the other hand, $(yx)^2 = yxyx = (xyb)(xyb) = (xy)^2b^2 = bcb^2 = c$. It follows that $b^2 = 1$-contradiction to the hypothesis (b is an element of order 3).

3) $x^2 = a$ and $y^2 = a^2$. Then $(yx)^2 = yxyx = y(xy)x = y(yxb^2)x = y^2x^2b^2 = a^2ab^2 = b^2$ and $(xy)^2 = xyxy = x(yx)y = x(xyb)y = x^2y^2b = b$. On the other hand, $(yx)^2 = yxyx = (xyb)(xyb) = (xy)^2b^2 = bb^2 = 1$. It follows that $b^2 = 1$-contradiction to the hypothesis.

4) $x^2 = y^2 = c \in \{a, a^2\}$. Then $(yx)^2 = yxyx = y(xy)x = y(yxb^2)x = y^2x^2b^2 = b^2c^2$ and $(xy)^2 = xyxy = x(yx)y = x(xyb)y = x^2y^2b = bc^2$. On the other hand, $(yx)^2 = yxyx = (xyb)(xyb) = (xy)^2b^2 = bc^2b^2 = c^2$. It follows that $b^2 = 1$-contradiction to the hypothesis.

So there is no group G of order 12 with $|Z(G)| = 3$.

Case V: $|Z(G)| = 2$. Then $|G/Z(G)| = 6$ and according to [9,4.2] either $G/Z(G) \cong \mathbf{Z}_6$ or $G/Z(G) \cong D_3$, where D_3 is the dihedral group of order 3. Since the group Z_6 is cyclic it follows that $G/Z(G) \cong D_3$.

Therefore $Z(G) = \langle a\rangle = \{1, a\}$, with $a \in G$ and $a^2 = 1$, and $G/Z(G) = \langle xZ(G), yZ(G)\rangle$, where x, $y \in G$, x^3, $y^2 \in Z(G)$ and $xyZ(G) = yx^2Z(G)$. Then $\{xy, xya\} = \{yx^2, yx^2a\}$. Now we have two subcases:

Subcase 1: $xy = yx^2$. Here we have the following possibilities:

i) $x^3 = 1 = y^2$. Then $yx = x^2y$ and there is a group $G_3 = \langle x, y, a\rangle$, with x, y and a satisfying the above conditions; see Table 3.

ii) $x^3 = 1$ and $y^2 = a$. Then again $yx = x^2y$ and there is a group $G_4 = \langle x, y\rangle$, with x and y satisfying the above conditions; see Table 4.

iii) $x^3 = a$ and $y^2 = 1$. Then we obtain that $x^6 = 1$ and $y = x^5yx^2$. It follows that $yx = x^5yx^3 = x^2y$ and $yx^2 = x^2yx = x^2x^2y = x^4y$. So $xy = x^4y$ and $x^3 = 1$-which is impossible. Therefore there is no group G with the above properties.

iv) $x^3 = a = y^2$. Then $x^6 = y^3 = 1$ and again we obtain that $yx = x^2y$ and $xy = x^4y$, which is impossible. So, also in these conditions there is no exist group G.

Subcase 2: $xy = yx^2a$. Also in this subcase we have the following possibilities:

i) $x^3 = 1 = y^2$. Then $yx = x^2ya$, $yx^2 = x^2yxa = x^2x^2yaa = xy$. So, $a = 1$-which is impossible. Therefore in these conditions there is no exist group G.

ii) $x^3 = 1$ and $y^2 = a$. Then $y^4 = 1$ and we obtain that $yx = x^2ya = x^2y^3$ and $yx^2 = (yx)x = x^2y^3x = (x^2y)(y^2x) = x^2(yx)y^2 = x^2x^2y^3y^2 = xy$. It follows that

$yx^2 = yx^2a$ and $a = y^2 = 1$-which is impossible. Therefore there is no exist group G with the above properties.

iii) $x^3 = a$ and $y^2 = 1$. Then we obtain that $x^6 = 1$ and $y = x^5yx^5$. It follows that $yx = x^5yx^6 = x^5y$ and there is a group $G_5 = \langle x, y\rangle$, with x and y satisfying the above conditions; see Table 5.

iv) $x^3 = a = y^2$. Then $x^6 = y^3 = 1$ and again we obtain that $yx = x^5y$ and there is a group $G_6 = \langle x, y\rangle$, with x and y satisfying the above conditions; see Table 6.

Case VI: $|Z(G)| = 1$. Then according to [8,5.11] G has two subgroups H and K such that $|H| = 4$ and $|K| = 3$. Assume $K = \langle y\rangle = \{1, y, y^2\}$, with $y \in G$. Since H is a commutative (sub)group of order 4 we have two subcases:

Subcase 1: $H \cong \mathbf{Z}_4$. In this subcase there is an element $x \in G$ such that $H = \langle x\rangle = \{1, x, x^2, x^3\}$. Since $Z(G) = \{1\}$ it follows that $H \cap K = \{1\}$. Then $HK = \{1, x, x^2, x^3, y, y^2, xy, xy^2, x^2y, x^2y^2, x^3y, x^3y^2\}$ is a set with 12 distinct elements from G. It follows that $G = HK$ and $HK = KH$. Hence $yx \in \{x^2y, x^3y, xy^2, x^2y^2, x^3y^2\}$ and in this subcase we have the following five possibilities:

i) $yx = x^2y$. Then $yx^2 = (yx)x = (x^2y)x = x^2x^2y = y$ and $x^2 = 1$-which is impossible.

ii) $yx = x^3y$. Then $yx^2 = x^3(yx) = x^3x^3y = x^2y$ and $y^2x^2 = y(yx^2) = yx^2y = x^2y^2$. So $x^2 \in Z(G)$, contradiction to the hypothesis.

iii) $yx = xy^2$. Again we obtain that $x^2 \in Z(G)$. Since the proof is similar as above, this will be left to the reader.

iv) $yx = x^2y^2$. Then $yx^2 = x^2y^2x = x^2yx^2y^2 = x^2x^2y^2xy^2 = y^2xy^2 = yx^2y$ and, so $y = 1$-which is impossible.

v) $yx = x^3y^2$. Then $xy = y^2x^3$ and $yx^2 = x^3y^2x = x^3yx^3y^2 = (x^2y^2)^2$. Since the element yx^2 belongs to $\{xy^2, x^2y, x^2y^2, x^3y\}$ we will study all these possibilities:

a) $yx^2 = x^2y$. Then $x^2y^2x^2y^2 = x^2y$, $y^2x^2y = 1$, $y^2x^2 = y^2$ and, so $x^2 = 1$-which is impossible.

b) $yx^2 = x^3y$. Then $x^3y^2x = x^3y$ and $yx = 1$-which is impossible.

c) $yx^2 = xy^2$. Then $x^3y^2x = xy^2$, $x^2y^2x = y^2$, $x^2y^2 = y^2x^3$ and, so $xy = 1$-which is impossible.

d) $yx^2 = x^2y^2$. Then $(x^2y^2)^2 = x^2y^2$ and, so $x^2y^2 = 1$-which is impossible.

Otherwise: $(x^2y^2)^2 = (xyx)^2 = (xy)(x^2y)x = (xyx^2)(x^3y^2) = (xy)(xy^2) = x(x^3y^2)y^2 = y$. On the other hand, $(x^2y^2)^2 = x^2(xyxy^2) = x^3(yx)y^2 = x^3(x^3y^2y^2) = x^2y$. It follows that $x^2 = 1$-which is impossible.

Therefore in the conditions from this subcase there is no group G of order 12.

Subcase 2: $H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. In this subcase there are two elements $x, z \in G$ such that $H = \langle x, z\rangle = \{1, x, z, xz = zx\}$ and $x^2 = z^2 = (xz)^2 = 1$. Let n_2 be the number of the 2-sylow subgroups of G and n_3 the number of the 3-sylow subgroups of G. Then according to [5,3.7 (p. 145)] (the third Sylow's theorem) n_2 divide 3, n_3 divide 4, $n_2 \equiv 1(mod\ 2)$ and $n_3 \equiv 1(mod\ 3)$.

1) If $n_3 = 1$ then according to [5,4.2 (p. 146)] G has only two elements of order 3; these are y and y^2. It follows that G has 10 elements which there aren't of order 3.

i) If $n_2 = 1$ then G has only one 2-sylow subgroup, namely H and also in G there are 6 elements of order 6; so $o(xy) = 6$. Then since $Z(G) = 1$ it follows that $xy \neq yx$, $zy \neq yz$ and the following sets $A_x = \{x, y^2xy, yxy^2\}$, $A_z = \{z, y^2zy, yzy^2\}$, $A_y = \{y, xyx, zyz\}$ and $A_{y^2} = \{y^2, xy^2x, zy^2z\}$ have distinct elements. If for a $g \in G$ denote by n_g the cardinal of the set $\{a^{-1}ga \mid a \in G\}$ then: $3 = |A_x| \leq |n_x|$, $3 = |A_z| \leq |n_z|$, $3 = |A_y| \leq |n_y|$ and $3 = |A_{y^2}| \leq |n_{y^2}|$. By the class equation: $|G| = |Z(G)| + \sum\limits_{a \in G \setminus Z(G)} n_a$, in our case, obtain that $12 \geq 1 + 3 + 3 + 3 + 3 = 13$-impossible.

ii) If $n_2 = 3$ then G has three distinct 2-sylow subgroups; say H_1, H_2, H_3, each isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. It follows that there are $a \in H_1 \setminus H_2$, $b \in H_2 \setminus H_3$ and $c \in H_3 \setminus H_1$ such that $a^2 = b^2 = c^2 = 1$.

If $H_1 \cap H_2 = H_2 \cap H_3 = H_3 \cap H_1 = 1$ (1) then G has 9 elements of order 2. It follows that $\text{ord}(ab) = 2$ and, so $ab = ba \in H_1 \cup H_2 \cup H_3$. We study these possibilities:

a) If $ab = a' \in H_1$ then $b = aa' \in H_1 \cap H_2$-contradicting (1).

b) If $ab = b' \in H_2$ then $a = bb' \in H_1 \cap H_2$, again contradicting (1).

c) If $ab = c' \in H_3$ then $b = ac' \in H_2$ and $ac' = c'a$. Hence the subgroup $L = \{1, a, b, c'\}$ is contained in $\{H_1, H_2, H_3\}$, also -contradicting (1).

It follows that $H_1 \cap H_2 = \{1, a'\}$. Then $H_1 = \{1, a', x', a'x'\}$ and $H_2 = \{1, a', z', a'z'\}$. It follows that $x'z'$, $a'x'z' \in G \setminus (H_1 \cup H_2)$ and $\text{ord}(x'z') = \text{ord}(a'x'z') \in \{2, 6\}$, because $(x'z')^k = (a'x'z')^k$ for every $k \in \{2, 4, 6\}$.

If $\text{ord}(x'z') = \text{ord}(a'x'z') = 6$ then $(x'z')^2$, $(a'x'z')^2 \in \{y, y^2\}$. It follows that $(x'z')^4 = a'x'z'$ and $a' = (x'z')^3$. But, $(a'x'z')^4 = x'z'$ and $(a'x'z')^3 = a'$. So, $(x'z')^3 = (a'x'z')^3$, and since $(x'z')^2 = (a'x'z')^2$ it follows that $a' = 1$, which is impossible. So $\text{ord}(x'z') = \text{ord}(a'x'z') = 2$, $x'z' = z'x' \in H_3 \setminus (H_1 \cup H_2)$, and $a'x'z' = z'x'a' \in H_3 \setminus (H_1 \cup H_2)$; hence $H_3 = \{1, x'z', a'x'z', a'\}$. Then $M = \{1, x', z', a', x'z', a'x', a'z', a'x'z'\}$ is a commutative subgroup of order 8 of G (see the following table), which is impossible. (On observe that M is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.)

The table of (sub)group M**:** $M = < 1, x', z', a' >$**,** $x'z' = z'x'$**,** $x'a' = a'x'$**,** $z'a' = a'z'$**,** $(x')^2 = (z')^2 = (a')^2 = 1$

$\cdot$	1	x'	z'	a'	$x'z'$	$x'a'$	$z'a'$	$x'z'a'$
1	1	x'	z'	a'	$x'z'$	$x'a'$	$z'a'$	$x'z'a'$
x'	x'	1	$x'z'$	$x'a'$	z'	a'	$x'z'a'$	$z'a'$
z'	z'	$x'z'$	1	$z'a'$	x'	$x'z'a'$	a'	$x'a'$
a'	a'	$x'a'$	$z'a'$	1	$x'z'a'$	x'	z'	$x'z'$
$x'z'$	$x'z'$	z'	x'	$x'z'a'$	1	$z'a'$	$x'a'$	a'
$x'a'$	$x'a'$	a'	$x'z'a'$	x'	$z'a'$	1	$x'z'$	z'
$z'a'$	$z'a'$	$x'z'a'$	a'	z'	$x'a'$	$x'z'$	1	x'
$x'z'a'$	$x'z'a'$	$z'a'$	$x'a'$	$x'z'$	a'	z'	x'	1

2) If $n_3 > 1$ then either $n_3 = 2$ or $n_3 = 4$. In the first case by [5,3.7 (p. 145)] obtain that $2 \equiv 1$ (mod3)-impossible. Therefore $n_3 = 4$ and the group G has 8 elements of order 3-see again [5,4.2 (p. 146)]. It follows that G has only one

2-sylow subgroup, namely $H = \langle x, z\rangle = \{1, x, z, xz = zx\}$, with $x,\ z \in G$ and $x^2 = z^2 = (xz)^2 = 1$. Then $G = \{1, x, z, y, y^2, xz, xy, xy^2, zy, zy^2, xzy, xzy^2$. Since $\mathrm{ord}(xy) = 3$ it follows that $(xy)^2 = y^2x$. But $y^2x \in \{xy^2, zy, xz, zy^2, xzy, xzy^2\}$. We study all possibilities:

i) $y^2x = xy^2$. Then $(xy)^2 = xyy$ and $x = 1$-impossible.

ii) $y^2x = zy$. Then $(xy)^2 = zy$, $xyx = z$, $xy^2x = 1$ and $y^2 = 1$-impossible.

iii) $y^2x = xz$. Then $y^2 = xzx = z$-impossible.

iv) $y^2x = zy^2$. Then $yx = xzy$, $yz = xy$ and there is a group G_7 with these properties-see Table 7.

v) $y^2x = xzy$. Then $(xy)^2 = zxy$ and $xy = z$-impossible.

vi) $y^2x = xzy^2$. Then $yx = zy$, $yz = xzy$ and there is a group G_8 with these properties-see Table 8.

Therefore we have determined 8 groups of order 12. Afterwards we are going to show the following isomorphisms: $G_3 \cong G_5$, $G_4 \cong G_6$ and $G_7 \cong G_8$. First we make up the following tables:

The group	The elements of order 2	The elements of order 3
G_1	x^6	x^4, x^8
G_2	x, y, xy	z, z^2
G_3	$y, a, xy, x^2y, ya, xya, x^2ya$	x, x^2
G_4	y^2	x, x^2
G_5	$x^3, y, xy, x^2y, x^3y, x^4y, x^5y$	x^2, x^4
G_6	x^3	x^2, x^4
G_7	x, z, xz	$y, y^2, xy, xy^2, zy, zy^2, xzy, xzy^2$
G_8	x, z, xz	$y, y^2, xy, xy^2, zy, zy^2, xzy, xzy^2$

The group	The elements of order 4	The elements of order 6
G_1	x^3, x^9	x^2, x^{10}
G_2	-	$xz, xz^2, yz, yz^2, xyz, xyz^2$
G_3	-	xa, x^2a
G_4	$y, y^2, xy, xy^3, x^2y, x^2y^3$	xy^2, x^2y^2
G_5	-	x, x^5
G_6	$y, xy, x^2y, x^3y, x^4y, x^5y$	x, x^5
G_7	-	-
G_8	-	-

Now, it is straightforward to verify that:

α) the map $f : G_3 \to G_5$ defined by $f(a) = x^3$, $f(y) = y$ and $f(x) = x^4$ is an isomorphism of groups and $G_3 \cong G_5$;

β) the map $f : G_4 \to G_6$ defined by $f(x) = x^4$, $f(y) = y$ and $f(y^2) = x^3$ is an isomorphism of groups and $G_4 \cong G_6$;

γ) the map $f : G_7 \to G_8$ defined by $f(x) = z$, $f(z) = x$ and $f(y) = y$ is an isomorphism of groups and $G_7 \cong G_8$.

Now the theorem is completely proved.

Corollary 1: *The group G_3 is isomorphic to $\mathbf{Z}_2 \times S_3$, where S_3 is the symmetric group of degree* 3.

Proof: In the group G_3, the set $A = \{1, x, x^2, y, xy, x^2y\}$ is a non-commutative subgroup of order 6. So, A is isomorphic to S_3.

Corollary 2: *The group G_4 is isomorphic to a semidirect product of $\mathbf{Z}_3$ by $\mathbf{Z}_4$.*

Proof: Left to the reader.

Corollary 3: *The group G_8 is isomorphic to A_4-the alternating group of degree* 4.

Proof: The 12 elements of A_4 are eight 3-cycles, three products of disjoint transpositions, and the identity. These elements are: $\sigma = (1\ 2\ 3)$, $\sigma^2 = (1\ 3\ 2)$, $\tau = (1\ 4)(2\ 3)$, $\mu = (1\ 2)(3\ 4)$, $\tau\mu = \mu\tau = (1\ 3)(2\ 4)$, $e = (1)$, $\tau\sigma = (1\ 3\ 4)$, $\mu\sigma = (2\ 4\ 3)$, $\tau\sigma^2 = (1\ 2\ 4)$, $\mu\sigma^2 = (1\ 4\ 3)$, $\tau\mu\sigma = (1\ 4\ 2)$ and $\tau\mu\sigma^2 = (2\ 3\ 4)$. Now, it is easy to check that the map $f : G_8 \to A_4$, defined by: $f(x) = \tau$, $f(y) = \sigma$ and $f(z) = \mu$ is an isomorphism of groups.

Corollary 4: *The group G_8 is generated by the elements y, xy and zy.*

Proof: It is straightforward to verify that $A_4 =< \sigma, \tau\sigma, \mu\sigma >$. Now, Corollary 3 completes the proof.

Corollary 5: *The group G_8 does not have no subgroup of order* 6.

Proof: From [8,3.11] it follows that A_4 is a group of order 12 having no subgroup of order 6. Again Corollary 3 completes the proof.

Corollary 6: *The group G_8 is not simple.*

Proof: The set $V = \{\tau, \mu, \tau\mu = \mu\tau\}$ it is easily seen to be a subgroup of A_4. Since V contain all the permutations of S_4 of a given cycle structure, V is normal in S_4, a fortiori, it is normal in A_4. Therefore A_4 is not simple, and according to Corollary 3, G_8 has the same property. (On observe that V is isomorphic to H).

Otherwise: It is easy to check that the subgroup $H =< x, z >$ is normal in G_8.

Corollary 7: *The normality of subgroups of a group G need not be transitive.*

Proof: Counterexample: in G_8, the subgroup $N = \{1, xz\}$ is normal in $H = \{1, x, z, xz\}$, which is normal in G_8 (see Corollary 6), but since $yN \neq Ny$ it follows that N is not normal in G_8.

Corollary 8: *Every group G of order* 12 *that is not isomorphic to A_4 contains an element of order* 6*-[8,5.16].*

Proof: The elements: $x \in G_1$, $yz \in G_2$, $xa \in G_3$, $xy^2 \in G_4$, $x \in G_5$, $x \in G_6$ are of order 6.

The (multiplication) tables of these 8 groups which have been determined. Table 1: $G_1 \cong \mathbf{Z}_{12}$

$\cdot$	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}
1	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}
x	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	1
x^2	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	1	x
x^3	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	1	x	x^2
x^4	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	1	x	x^2	x^3
x^5	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	1	x	x^2	x^3	x^4
x^6	x^6	x^7	x^8	x^9	x^{10}	x^{11}	1	x	x^2	x^3	x^4	x^5
x^7	x^7	x^8	x^9	x^{10}	x^{11}	1	x	x^2	x^3	x^4	x^5	x^6
x^8	x^8	x^9	x^{10}	x^{11}	1	x	x^2	x^3	x^4	x^5	x^6	x^7
x^9	x^9	x^{10}	x^{11}	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
x^{10}	x^{10}	x^{11}	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
x^{11}	x^{11}	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}

Table 2: $G_2 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_3$

$\cdot$	1	x	y	z	z^2	xy	xz	xz^2	yz	yz^2	xyz	xyz^2
1	1	x	y	z	z^2	xy	xz	xz^2	yz	yz^2	xyz	xyz^2
x	x	1	xy	xz	xz^2	y	z	z^2	xyz	xyz^2	yz	yz^2
y	y	xy	1	yz	yz^2	x	xyz	xyz^2	z	z^2	xz	xz^2
z	z	xz	yz	z^2	1	xyz	xz^2	x	yz^2	y	xyz^2	xy
z^2	z^2	xz^2	yz^2	1	z	xyz^2	x	xz	y	yz	xy	xyz
xy	xy	y	x	xyz	xyz^2	1	yz	yz^2	xz	xz^2	z	z^2
xz	xz	z	xyz	xz^2	x	yz	z^2	1	xyz^2	xy	yz^2	y
xz^2	xz^2	z^2	xyz^2	x	xz	yz^2	1	z	yz^2	xyz	y	yz
yz	yz	xyz	z	yz^2	y	xz	xyz^2	xy	z^2	1	xz^2	x
yz^2	yz^2	xyz^2	z^2	y	yz	xz^2	xy	xyz	1	z	x	xz
xyz	xyz	yz	xz	xyz^2	xy	z	yz^2	y	xz^2	x	z^2	1
xyz^2	xyz^2	yz^2	xz^2	xy	xyz	z^2	y	yz	x	xz	1	z

Table 3: $G_3 =< x, y, a >$, $x^3 = y^2 = a^2 = 1$, $yx = x^2y$, $Z(G) =< a >$

$\cdot$	1	x	x^2	y	a	xy	x^2y	xa	x^2a	ya	xya	x^2ya
1	1	x	x^2	y	a	xy	x^2y	xa	x^2a	ya	xya	x^2ya
x	x	x^2	1	xy	xa	x^2y	y	x^2a	a	xya	x^2ya	ya
x^2	x^2	1	x	x^2y	x^2a	y	xy	a	xa	x^2ya	ya	xya
y	y	x^2y	xy	1	ya	x^2	x	x^2ya	xya	a	x^2a	xa
a	a	xa	x^2a	ya	1	xya	x^2ya	x	x^2	y	xy	x^2y
xy	xy	y	x^2y	x	xya	1	x^2	ya	x^2ya	xa	a	x^2a
x^2y	x^2y	xy	y	x^2	x^2ya	x	1	xya	ya	x^2a	xa	a
xa	xa	x^2a	a	xya	x	x^2ya	ya	x^2	1	xy	x^2y	y
x^2a	x^2a	a	xa	x^2ya	x^2	ya	xya	1	x	x^2y	y	xy
ya	ya	x^2ya	xya	a	y	x^2a	xa	x^2y	xy	1	x^2	x
xya	xya	ya	x^2ya	xa	xy	a	x^2a	y	x^2y	x	1	x^2
x^2ya	x^2ya	xya	ya	x^2a	x^2y	xa	a	xy	y	x^2	x	1

Table 4: $G_4 =< x, y >$, $x^3 = 1$, $y^2 = a$, $yx = x^2y$, $Z(G) =< a >$

$\cdot$	1	x	x^2	y	y^2	y^3	xy	xy^2	xy^3	x^2y	x^2y^2	x^2y^3
1	1	x	x^2	y	y^2	y^3	xy	xy^2	xy^3	x^2y	x^2y^2	x^2y^3
x	x	x^2	1	xy	xy^2	xy^3	x^2y	x^2y^2	x^2y^3	y	y^2	y^3
x^2	x^2	1	x	x^2y	x^2y^2	x^2y^3	y	y^2	y^3	xy	xy^2	xy^3
y	y	x^2y	xy	y^2	y^3	1	x^2y^2	x^2y^3	x^2	xy^2	xy^3	x
y^2	y^2	xy^2	x^2y^2	y^3	1	y	xy^3	x	xy	x^2y^3	x^2	x^2y
y^3	y^3	x^2y^3	xy^3	1	y	y^2	x^2	x^2y	x^2y^2	x	xy	xy^2
xy	xy	y	x^2y	xy^2	xy^3	x	y^2	y^3	1	x^2y^2	x^2y^3	x^2
xy^2	xy^2	x^2y^2	y^2	xy^3	x	xy	x^2y^3	x^2	x^2y	y^3	1	y
xy^3	xy^3	y^3	x^2y^3	x	xy	xy^2	1	y	y^2	x^2	x^2y	x^2y^2
x^2y	x^2y	xy	y	x^2y^2	x^2y^3	x^2	xy^2	xy^3	x	y^2	y^3	1
x^2y^2	x^2y^2	y^2	xy^2	x^2y^3	x^2	x^2y	y^3	1	y	xy^3	x	xy
x^2y^3	x^2y^3	xy^3	y^3	x^2	x^2y	x^2y^2	x	xy	xy^2	1	y	y^2

Table 5: $G_6 =< x, y >$, $x^3 = a$, $y^2 = a^2 = 1$, $yx = x^5y$, $Z(G) =< a >$

$\cdot$	1	x	x^2	x^3	x^4	x^5	y	xy	x^2y	x^3y	x^4y	x^5y
1	1	x	x^2	x^3	x^4	x^5	y	xy	x^2y	x^3y	x^4y	x^5y
x	x	x^2	x^3	x^4	x^5	1	xy	x^2y	x^3y	x^4y	x^5y	y
x^2	x^2	x^3	x^4	x^5	1	x	x^2y	x^3y	x^4y	x^5y	y	xy
x^3	x^3	x^4	x^5	1	x	x^2	x^3y	x^4y	x^5y	y	xy	x^2y
x^4	x^4	x^5	1	x	x^2	x^3	x^4y	x^5y	y	xy	x^2y	x^3y
x^5	x^5	1	x	x^2	x^3	x^4	x^5y	y	xy	x^2y	x^3y	x^4y
y	y	x^5y	x^4y	x^3y	x^2y	xy	1	x^5	x^4	x^3	x^2	x
xy	xy	y	x^5y	x^4y	x^3y	x^2y	x	1	x^5	x^4	x^3	x^2
x^2y	x^2y	xy	y	x^5y	x^4y	x^3y	x^2	x	1	x^5	x^4	x^3
x^3y	x^3y	x^2y	xy	y	x^5y	x^4y	x^3	x^2	x	1	x^5	x^4
x^4y	x^4y	x^3y	x^2y	xy	y	x^5y	x^4	x^3	x^2	x	1	x^5
x^5y	x^5y	x^4y	x^3y	x^2y	xy	y	x^5	x^4	x^3	x^2	x	1

Table 6: $G_6 =< x, y >$, $x^3 = y^2 = a$, $yx = x^5y$, $a^2 = 1$ $Z(G) =< a >$

$\cdot$	1	x	x^2	x^3	x^4	x^5	y	xy	x^2y	x^3y	x^4y	x^5y
1	1	x	x^2	x^3	x^4	x^5	y	xy	x^2y	x^3y	x^4y	x^5y
x	x	x^2	x^3	x^4	x^5	1	xy	x^2y	x^3y	x^4y	x^5y	y
x^2	x^2	x^3	x^4	x^5	1	x	x^2y	x^3y	x^4y	x^5y	y	xy
x^3	x^3	x^4	x^5	1	x	x^2	x^3y	x^4y	x^5y	y	xy	x^2y
x^4	x^4	x^5	1	x	x^2	x^3	x^4y	x^5y	y	xy	x^2y	x^3y
x^5	x^5	1	x	x^2	x^3	x^4	x^5y	y	xy	x^2y	x^3y	x^4y
y	y	x^5y	x^4y	x^3y	x^2y	xy	x^3	x^2	x	1	x^5	x^4
xy	xy	y	x^5y	x^4y	x^3y	x^2y	x^4	x^3	x^2	x	1	x^5
x^2y	x^2y	xy	y	x^5y	x^4y	x^3y	x^5	x^4	x^3	x^2	x	1
x^3y	x^3y	x^2y	xy	y	x^5y	x^4y	1	x^5	x^4	x^3	x^2	x
x^4y	x^4y	x^3y	x^2y	xy	y	x^5y	x	1	x^5	x^4	x^3	x^2
x^5y	x^5y	x^4y	x^3y	x^2y	xy	y	x^2	x	1	x^5	x^4	x^3

Table 7: $G_7 =< x, y, z >$, $x^2 = y^3 = z^2 = 1$, $yx = xzy$, $xz = zx$, $yz = xy$,
$Z(G) = \{1\}$

$\cdot$	1	x	z	xz	y	y^2	xy	xy^2	zy	zy^2	xzy	xzy^2
1	1	x	z	xz	y	y^2	xy	xy^2	zy	zy^2	xzy	xzy^2
x	x	1	xz	z	xy	xy^2	y	y^2	xzy	xzy^2	zy	zy^2
z	z	xz	1	x	zy	zy^2	xzy	xzy^2	y	y^2	xy	xy^2
xz	xz	z	x	1	xzy	xzy^2	zy	zy^2	xy	xy^2	y	y^2
y	y	xzy	xy	zy	y^2	1	xzy^2	xz	xy^2	x	zy^2	z
y^2	y^2	zy^2	xzy^2	xy^2	1	y	z	zy	xz	xzy	x	xy
xy	xy	zy	y	xzy	xy^2	x	zy^2	z	y^2	1	xzy^2	xz
xy^2	xy^2	xzy^2	zy^2	y^2	x	xy	xz	xzy	z	zy	1	y
zy	zy	xy	xzy	y	zy^2	z	xy^2	x	xzy^2	xz	y^2	1
zy^2	zy^2	y^2	xy^2	xzy^2	z	zy	1	y	x	xy	xz	xzy
xzy	xzy	y	zy	xy	xzy^2	xz	y^2	1	zy^2	z	xy^2	x
xzy^2	xzy^2	xy^2	y^2	zy^2	xz	xzy	x	xy	1	y	z	zy

Table 8: $G_8 =< x, y, z >$, $x^2 = y^3 = z^2 = 1$, $yx = zy$, $xz = zx$, $yz = xzy$, $Z(G) = \{1\}$

$\cdot$	1	x	z	xz	y	y^2	xy	xy^2	zy	zy^2	xzy	xzy^2
1	1	x	z	xz	y	y^2	xy	xy^2	zy	zy^2	xzy	xzy^2
x	x	1	xz	z	xy	xy^2	y	y^2	xzy	xzy^2	zy	zy^2
z	z	xz	1	x	zy	zy^2	xzy	xzy^2	y	y^2	xy	xy^2
xz	xz	z	x	1	xzy	xzy^2	zy	zy^2	xy	xy^2	y	y^2
y	y	zy	xzy	xy	y^2	1	zy^2	z	xzy^2	xz	xy^2	x
y^2	y^2	xzy^2	xy^2	zy^2	1	y	xz	xzy	x	xy	z	zy
xy	xy	xzy	zy	y	xy^2	x	xzy^2	xz	zy^2	z	y^2	1
xy^2	xy^2	zy^2	y^2	xzy^2	x	xy	z	zy	1	y	xz	xzy
zy	zy	y	xy	xzy	zy^2	z	y^2	1	xy^2	x	xzy^2	xz
zy^2	zy^2	xy^2	xzy^2	y^2	z	zy	x	xy	xz	xzy	1	y
xzy	xzy	xy	y	zy	xzy^2	xz	xy^2	x	y^2	1	zy^2	z
xzy^2	xzy^2	y^2	zy^2	xy^2	xz	xzy	1	y	z	zy	x	xy

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