CREATIVE MATH.
14 (2005), 83 - 86

Analysis of duration and convexity of coupon obligation

Vincent Šoltés and Michal Šoltés

Abstract. Duration and convexity of coupon bonds are analysed in this paper. There is derived new formula for duration of the coupon bonds. The proof is based on the calculation of the sum some special sequences without using derivation and integration.

1. Introduction

The main objective of the paper is to analyze duration and convexity of coupon obligation. It is just this analysis that will help us understand their definitions and prove the statement applying these characteristics in certain sense.

If coupon obligation with nominal value F, coupon rate r, expiration period n and desired return rate i is supposed, then theoretical value of the obligation can be calculated by well known formula:

$$V = \sum_{t=1}^{n} \frac{Fr}{(1+i)^t} + \frac{F}{(1+i)^n} \quad . \tag{1}$$

Duration D of such obligation is defined as follows:

$$D = \frac{\sum\limits_{t=1}^{n} t \cdot \dfrac{Fr}{(1+i)^t} + n \cdot \dfrac{F}{(1+i)^n}}{V} \tag{2}$$

and convexity with the formula

$$K = \frac{1}{(1+i)^2} \left[\sum_{t=1}^{n} \frac{t\,(t+1)\,Fr}{(1+i)^t} + \frac{n\,(n+1) \cdot F}{(1+i)^n} \right]. \tag{3}$$

Relation (1) is clear, as, it is, actually, the present value of cash flows from the obligation that is the same as theoretical value of obligation. It is obvious as well that the higher interest rate is, the lower theoretical price of obligation is and vice-versa, the lower interest rate is, the higher theoretical price of obligation is.

Relations (2) and (3) are taken as a matter of course by definition. Nevertheless, it is interesting to find out how these definitions came out and why it was important to define such terms. Economy literature shows statements, which say that duration and convexity are the criteria of price of obligation sensitivity to the interest rate variation. The higher duration or convexity is, the more sensitive obligation to its

Received: 01.09.2004. In revised form: 30.11.2004.
2000 *Mathematics Subject Classification.* 91B28, 91B26, 91B24.
Key words and phrases. *Duration, convexity, obligation, price of obligation, Taylor´s formula.*

price at the interest rate variation is. But this cannot be seen from definitions (2) and (3).

2. Fundamentals of duration and convexity

Let us calculate relative change of theoretical price V at the change of interest rate i at first. We have

$$\frac{\frac{\partial V}{\partial i}}{V}=\frac{\sum\limits_{t=1}^{n}-t\cdot\frac{Fr}{(1+i)^{t+1}}-n\cdot\frac{F}{(1+i)^{n+1}}}{V}=-\frac{1}{1+i}\left[\frac{\sum\limits_{t=1}^{n}t\cdot\frac{Fr}{(1+i)^{t}}+n\cdot\frac{F}{(1+i)^{n}}}{V}\right] \quad ,$$

then using (2) we easily find out, that

$$D=-\frac{\frac{\partial V}{\partial i}\cdot(1+i)}{V}\,. \tag{4}$$

Let us calculate the second partial derivation of theoretical price V by interest rate i. We have

$$\frac{\partial^2 V}{\partial i^2}=\frac{\partial}{\partial i}\left[\sum_{t=1}^{n}-\frac{t\cdot Fr}{(1+i)^{t+1}}-\frac{n\cdot F}{(1+i)^{n+1}}\right]=\sum_{t=1}^{n}\frac{t(t+1)\cdot Fr}{(1+i)^{t+2}}+\frac{n(n+1)\cdot t}{(1+i)^{n+2}}=$$

$$=\frac{1}{(1+i)^2}\left[\sum_{t=1}^{n}\frac{t(t+1)\cdot Fr}{(1+i)^{t}}+\frac{n(n+1)\cdot F}{(1+i)^{n}}\right]$$

so

$$K=\frac{\partial^2 V}{\partial i^2} \quad . \tag{5}$$

We can see, that convexity of coupon obligation is simply the second partial derivation of its theoretical price by interest rate.

3. Usage of duration and convexity for calculation of estimation of theoretical price change at the change of interest price

Let us suppose coupon obligation, we know its theoretical price V_1, duration D and convexity K at the interest rate i_1. The aim is to calculate, how its theoretical price will change when interest rate changes from value i_1 to value i_2. Using Taylor's sequence we obtain, that this price V_2 can be calculated by formula

$$V_2=V_1+\sum_{n=1}^{\infty}\frac{\left(\frac{\partial^n V}{\partial i^n}\right)_{i=i_1}\cdot(i_2-i_1)^n}{n!} \quad . \tag{6}$$

If we mark change of theoretical price $\Delta V=V_2-V_1$, change of interest rate $\Delta i=i_2-i_1$ and from Taylor's sequence we suppose only items with first and second partial derivation, then using (4) and (5) we obtain, that

$$\Delta V \doteq -\frac{DV_1}{1+i_1} \cdot \Delta i + \frac{1}{2} K (\Delta i)^2 \quad . \tag{7}$$

Relationship (7) allows us to estimate the change of theoretical price of obligation at the change of interest rate i, if we know duration and convexity. We can observe continual proportion; i. e. the change is higher with higher duration D or convexity K. As weight of duration in the relationship (7) $p_1 = -\frac{V_1 \cdot \Delta i}{1+i_1}$ is significantly higher than weight of convexity $p_2 = \frac{1}{2} (\Delta i)^2$, duration only is used in practice. Nevertheless, it is clear that both are very important characteristics, which should be known to every investor when investing to obligations.

4. Derivation of new formulas for calculating convexity and duration

Main disadvantage of formulas (2) and (3) is the fact, that they include sum with respect to maturity period of obligation and calculation may be long and difficult at the high maturity period. In the paper [4], a new formula for calculation of duration of coupon obligation is derived using integration of power sequences and subsequent derivation of its sum. In this part, we derive this formula using only elementary mathematics.

From formulas (2) and (1) after using substitution $x = \frac{1}{1+i}$ we have

$$D = \frac{\sum\limits_{t=1}^{n} t \cdot x^t + \frac{n}{r} \cdot x^n}{\sum\limits_{t=1}^{n} x^t + \frac{1}{r} \cdot x^n}.$$

Firstly, we find the sum of series in the numerator. We have

$$\begin{aligned}
&\sum_{t=1}^{n} t \cdot x^t = x + 2x^2 + 3x^3 + ... + nx^n = \\
&= \left(x + x^2 + x^3 + ... + x^n\right) + \left(x^2 + x^3 + ... + x^n\right) + ... + \left(x^{n-1} + x^n\right) + x^n = \\
&= x \cdot \frac{1-x^n}{1-x} + x^2 \cdot \frac{1-x^{n-1}}{1-x} \\
&+ ... + x^{n-1} \cdot \frac{1-x^2}{1-x} + x^n \cdot \frac{1-x}{1-x} = \\
&= \frac{x}{1-x} \left(1 - x^n + x - x^n + x^2 - x^n + ... + x^{n-1} - x^n\right) = \\
&= \frac{x}{1-x} \left(\frac{1-x^n}{1-x} - nx^n\right) = \\
&= \frac{x}{(1-x)^2} \left(1 - (n+1) \cdot x^n + nx^{n+1}\right)
\end{aligned}$$

As

$$\sum_{t=1}^{n} x^t = x \cdot \frac{1-x^n}{1-x},$$

after substitution for both sums of series, we have

$$D=\frac{\dfrac{x}{(1-x)^2}\left[1-(n+1)\,x^n+nx^{n+1}\right]+\dfrac{n}{r}\cdot x^n}{x\cdot\dfrac{1-x^n}{1-x}+\dfrac{1}{r}\cdot x^n},$$

from which after substitution for x and after modification we obtain

$$D=\frac{1}{i}\left[\frac{(1+i)^{n+1}+ni\left(\dfrac{i}{r}-1\right)-(1+i)}{(1+i)^n+\dfrac{i}{r}-1}\right] \tag{8}$$

which is the same as the formula (*) in the paper [4]. Formula (8) with the formula for convexity

$$K=\frac{Fr}{i^3}\left[\frac{2\,(1+i)^{n+2}-(n+1)\,(n+2)\,i^2-2\,(n+2)\,i-2+n\,(n+1)\,\dfrac{i^3}{r}}{(1+i)^{n+2}}\right] \tag{9}$$

derived in the paper [5] makes calculations of duration and convexity of coupon obligations much easier.

References

[1] Berinde, V., *Exploring, Investigating and Discovering in Mathematics*, Birkhäuser Verlag, 2004

[2] Chovancová, B., Jankovská, A., Kotlebová, J., Šturc, B., *Finančný trh. Eurounion*, Bratislava, 2002

[3] Hrvoová, B., *Analýza finanèných trhov* Sprint v fra, 2001

[4] Šoltés, V., Šoltés, M., *Durácia kupónovej obligácie ako kritérium cenovej citlivosti obligácie vzhl'adom na zmenu úrokových sadzieb*, Ekonomický časopis, 2004, 1, 108-114

[5] Šoltés, V., Šoltés, M., *Výpocet konvexity kupónovej obligácie a jej využitie na odhad zmeny teoretickej ceny obligácie pri zmene úrokových sadzieb* Ekonomické rozhlady, 3, 2004, 300-305

[6] Lešišin, J., *Financie v trhovej ekonomike.* Košice, 2004

Vincent Šoltés
Faculty of Economics
Department of Finance
Letná 9, 042 00 Košice
Slovakia
E-mail address: vincent.soltes@tuke.sk

Michal Šoltés
Faculty of Economics
Department of Banking and Investment
Letná 9, 042 00 Košice
Slovakia
E-mail address: michal.soltes@tuke.sk