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Richard Dedekind's theory of sets

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ABSTRACT. In this paper a short outline of Richard Dedekind's theory of sets is given.

George Cantor is considered to be the founder of the theory of sets. In a number of his works, especially articles from the 1890s he introduced many terms essential for this theory and developed it into a complex science (cf. [1]).

Cantor was not, however, the only mathematician of his times that noticed the importance of the theory of sets for mathematics. Some other scientists understood the need for a new strong foundation for mathematics. One of them was Richard Dedekind, who in the introduction to his small but important book *Was sind und was sollen die Zahlen?* made an explicit statement that logic was the appropriate foundation for mathematics (cf. [2]; also [4], p. 81–116, 218 – 223 and 241 – 256). He claimed that arithmetic, algebra or analysis were only parts of logic (*Teiles der Logik*), however logic that was understood much wider than it is today.

Having read *Was sind und was sollen die Zahlen?*, one can easily see that the construction of natural numbers, presented by Dedekind and being a complement to the programme of reducing analysis to arithmetic of natural numbers formulated earlier by Dirichlet, is based on set theory.

Before the content of Dedekind's book is presented, it is worth taking a closer look at his philosophical ideas. Although he did not describe himself as a follower of some philosophical school, it is possible to identify his philosophical *credo*, which he remained faithful to, on the basis on his statements.

As most German thinkers of the 19$^{\text{th}}$ century, Dedekind was heavily influenced by Kant. It does not mean, however, that he accepted Kant's beliefs without reservations, but more that he was immersed in a philosophical space, defined by Kant. Dedekind rejected Kant's concept of view (*Anschauung*) and *a priori* forms of sensuality (*Sinnlichkeit*), thanks to which an object is formed. It is not surprising though, as non-Euclidean geometry created half a century before had undermined the philosophy of intellectual cognition, created by Kant. Dedekind did not reject, however, the concept of mathematical *a priori*, but moved it to the intellectual sphere (*Verstand*). Mathematical concepts should therefore be created, according to Dedekind, with the help of *a priori* intellect functions, which just like intellect categories in Kant's presentation, were to synthesise the objects of cognition, presenting values available to our view as new units. So the foundations of mathematics should be sought in transcendental logic, which, according to Kant, should

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deal with *a priori* intellect functions. Interestingly enough, Kant included quantitative concepts such as *unit(y)*, *multitude* and *entirety* to the intellect, and not to the sensuality.

Dedekind did not fully develop his general philosophical theories; for mathematics, however, he was willing to base it on broadly understood logic. Contrary to Peirce, it was quite natural for him to include set theory in logic. This can be explained by the fact that Dedekind considered ranges of notions and ranges of terms as natural objects of logic. He was convinced at the same time that the whole mathematics could be reduced to logic, which makes him one of the main forerunners of logicism.

Was sind und was sollen die Zahlen? was published in 1888, but its first draft version was written between 1872 and 1878 and since then the ideas presented there had been familiar to scientists who were in close contact with Dedekind. As it can be seen, Dedekind's theory of sets came into being more or less at the time when Cantor's first works concerning theory of sets were published. Therefore, Dedekind's book can be considered as some version of this theory.

While Cantorian theory encompasses arithmetic of transfinite numbers and he seeks to justify his concepts concerning infinity (from the point of view close to Plato's), Dedekind limits his philosophical commentary considerably and his theory of sets includes only what is absolutely necessary to define arithmetic of natural numbers. It should be noted that Dedekind's presentation is much more modern in comparison with the one of Cantor. Theorems and commentary notes in Dedekind's book form logically ordered sequence and in this respect it can be compared to *The Elements* by Euclid, though it lacks axioms and postulates. The fundamentals of set theory are explained by reference to elementary, in the opinion of the author, intuitions connected with cognitive functions of the mind.

Let us have a closer look at Dedekind's book now. In the first chapter entitled *Systems of Elements*, he introduces fundamental concepts of set theory. In the first passage he writes:

> *An object (Ding) is anything one can think of. To talk about objects, they are designated by means of symbols, for example letters. One can talk about an object "a", or simply about "a", while in fact by "a" one understands a designated object, not just the letter "a". An object is entirely defined by everything one can say about it or think about it. An object "a" is the same as "b" (is identical with "b"), and "b" is the same as "a" if anything that can be thought of "a", can also be thought of "b", and everything that applies to "b" can also be thought of "a". The fact that "a" and "b" are only symbols or names of the same object shall be expressed by means of* $a = b$ *as well as* $b = a$. [...]. *If there is no such correspondence between objects designated as "a" and "b", then "a" and "b" are called different; "a" is a different object from "b" and "b" is a different object from "a", if there is some characteristic which describes one of these two objects and is not true about the other.* (cf. [2], p. 1)

In the next passage, Dedekind introduces the key terms for his theory:

> *It often happens that different objects a, b, c ..., for some reason considered from the common point of view, are put together in our mind; then one says that they form some system (System) S; objects a, b, c ... are called elements of this system S (die Elemente), if they are contained in S; or in other words S consists of these elements. Such a system (or entirety [Inbegriff, Gesamtheit] or manifold [Mannigfaltigkeit]) is itself an object, as it is something we can think of (cf. Section 1); S is completely defined if one can attribute each object to be (or not) an element of S. System S is therefore the same as system T, which is designated as S =T, if each element of S is also an element of T and each element of T is an element of S.* (cf. [2], p. 1–2).

As it can be seen, the term "system" is in fact equivalent to the term "set". And the above fragment can be considered as one of the first versions of a later formulated extensionality axiom.

In the next part of section 2, Dedekind allows for the existence of a system consisting of just one element (he does not seem to distinguish, however, between an object a and a system in which a is the only element) and at the same time does not consider an empty system, although, as he writes, *it can be convenient in some other considerations to take it into account.*

In section 3, in explanatory notes (*Erklärung*), the concept of a part of system (*Teil*) is described. A system A is called a part of a system S if each element A is also an element of S. In such a case Dedekind uses the following notation

$$A \prec S \quad \text{or} \quad S \succ A.$$

If A consists of only one element S, symbols are in fact identical: $s \prec S$. Let us add that Dedekind introduces the concept of the proper part in section 6 of the first chapter.

On the basis of definitions presented above, Dedekind formulates basic properties of the relation "$\prec$".

He states for example that

(i) $A \prec A$ (section 4)
(ii) $A \prec B$ and $B \prec A$, then $A = B$ (section 5)
(iii) $A \prec B$ and $B \prec C$, then $A \prec C$ (section 7).

The first chapter includes two more fundamental definitions: the sum of systems and the product of systems (section 8 and 17).

By the sum of systems $A, B, C, \dots$ (*zusammengezetzen System*), he understands a system $\mathfrak{M}(A, B, C...)$, consisting of the elements that belong to at least one of the systems $A, B, C, \dots$.

The product of systems $A, B, C\dots$ (*Gemeinheit der Systeme A,B,C,...*) is designated by means of $\mathfrak{G}(A, B, C...)$ and is understood as the biggest common part of systems $A, B, C, \dots$.

The second chapter of Dedekind's book is devoted to transformation of systems (on other systems). In the explanation included in passage 21, the author writes:

By ***transformation*** *(Abbildung) φ of a system S, we understand the law according to which for each defined element s from S there is a defined thing, called* ***an image*** *(Bild) s, designated as $\varphi(s)$* (cf. [2], p. 5).

It is typical that Dedekind considers it unnecessary to use a different letter as a symbol for range of transformation φ (the term "range" is not used). The range is completely defined by S and by φ; that is why it is designated as $\varphi(S)$. If the transformation φ is known, Dedekind uses s' and S' instead of $\varphi(s)$ and $\varphi(S)$.

Chapter 3 brings further fundamental definitions. In section 26, the author defines a similar transformation (*ähnlich*), or in other words, one-to-one transformation (*deutlich*) as the one in which different elements of system S are assigned to different images.

Similar transformations allow to define similar systems. Two systems R and S are similar if there is a similar transformation of system S to the system R, or in other words, a similar transformation φ for which $\varphi(S) = R$. In Cantorian terminology, similar systems are "equinumerable (*abzählbar*)" sets. Dedekind, unlike Cantor, does not develop the theory of similar systems, which would be analogous to the theory of power sets and the theory of cardinal numbers, formulated by Cantor. He notices, however, that all systems can be divided into classes (*Klassen*) consisting of systems similar to one another. Each class can be represented by some randomly chosen system.

The key concept for further discussion is the concept of a chain (*Kette*), introduced in Chapter 4. Dedekind considers here the transformation of a system into itself, that is transformation φ when $\varphi(S) \prec S$. A chain is any part K of a system S if $\varphi(K) \prec K$ (section 37). Obviously, whether K is or is not a chain depends on the type of transformation φ.

The main conclusion of Chapter 4 is the principle of complete induction. It is, however, different from the one that we commonly associate with natural numbers. Natural numbers have not been defined at all at this stage of considerations. Dedekind will do it later, formulating the principle of complete induction once again, this time for natural numbers.

Complete induction, as described in section 59, is presented in the language of Dedekind's set theory. It goes as follows:

If $\varphi_0(A)$ or alternatively A_0, if transformation φ is known, denotes the common part of all chains of a given system S, containing A ($\varphi_0(A)$ is a chain as well), then $\varphi_0(A)$ is a part of any system Σ included in S if and only if

(i) $A \prec \Sigma$
(ii) for any element s belonging at the same time to $\varphi_0(A)$ and Σ, $\varphi(s)$ belongs to Σ.

The above theorem allows us to decide if some property W is shared by all elements of a chain $\varphi_0(A)$. To do that, it is possible to prove that by complete induction

a) all elements of A have some property W
b) if an element a belonging to $\varphi_0(A)$ has the property W, then an element $\varphi(a)$ has this property as well.

As we will see, complete induction presented in this way is a strong foundation for arithmetic of natural numbers.

Chapter 5 of Dedekind's book is devoted to the presentation of finite and infinite systems. Section 64 contains a famous definition of an infinite system, which states that a system S is called infinite if it is in one-to-one correspondence with a proper part of itself. Otherwise, it is called a finite system. It goes without saying that Dedekind does not use an axiom of infinity, a part of Zermelo-Fraenkel set theory. If the existence of infinite systems was not the subject of the proper axiom, then it had to be proved in some way. In section 66, Dedekind states that infinite systems exist and then he proves it, just like Bolzano, by invoking the nature of the mind. (Bolzano at this point invoked the infinite nature of God's mind).

The proof: My universe of thoughts (meine Gedankenwelt), i.e. the entirety S of all objects I can think of, is infinite. And indeed if s is an element of S, then similarly the thought s' that s is an object of my thought, is itself an element of S. If s' is treated as an image $\varphi(s)$ of an element s, then the transformation φ of a system S is characterised by the fact that an image S' is a part of S; namely S' is the proper part of S, because there are elements in S (for instance my own I (mein eigenes Ich)) which are different from all other thoughts s' and that is why they do not belong to S'. Ultimately it becomes clear that if a, b are different elements of S, then their images a', b' are different as well, so the transformation φ similar (one-to-one). Consequently S is infinite; q.e.d. (cf. [2], p. 14).

From a modern point of view, the above "proof" is a confusion of some notions. It is, in fact, philosophical reasoning, which only uses some elements of mathematics.

Chapter 6 deals with so called simply infinite systems (*einfach unendlich*). These systems form the direct foundation on which Dedekind builds his arithmetic of natural numbers. A simply infinite system is defined as a system N, which when transformed in itself, is a chain $\varphi(1)$ of some element designated as 1, and which element does not belong to $\varphi(N)$. So N is a simply infinite system if:

(i) $\varphi(N) \prec N$,
(ii) $N = \varphi_0(1)$,
(iii) 1 does not belong to $\varphi(N)$,
(iv) transformation φ is similar.

In section 72 Dedekind shows that each infinite system contains a simply infinite system as its part. This statement corresponds to a well-known fact from Cantor's theory of cardinal numbers, which states that the smallest transfinite number is $\aleph_0$.

An explanatory note from section 73 of Chapter 6 is one of the most important in the whole book as it contains a definition of natural numbers. This definition is based on an important characteristic of simply infinite systems, which are, in fact, ordered by transformation φ.

Dedekind writes:

- *If considering a simply infinite system N, ordered by a transformation φ, one entirely ignores the specific nature of its elements, and takes into consideration only the fact that these elements are different from one another and that they correspond to one another by transformation φ, which orders them, then these elements shall be called* ***natural numbers****, or* ***ordinal numbers****, or simply* ***numbers*** (cf. [2], p. 17).

In this way a system of natural numbers in which transformation φ establishes some ordering has been defined. In this system natural numbers can be exchanged: $1, \varphi(1), \varphi(\varphi(1))$ etc.

As it can be seen, φ is in fact an equivalent to a successor in Peano's theory of natural numbers.

The next part of Dedekind's book, less interesting from our point of view because the author seizes to use precise terms from set theory, corresponds in many respects with Peano's concepts concerning arithmetic of natural numbers. In the next few chapters, Dedekind defines addition, multiplication and involution of natural numbers by means of induction; he also describes the relations of ordering in a set of natural numbers. In the last chapter, he considers finite systems and the number of elements in finite systems (*Anzahlen*).

Was sind und was sollen die Zahlen? is unquestionably the fullest presentation of Dedekind's set theory. Yet, his other works also contain elements of set theory.

A good example could be an article *Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teil* from 1897 (cf. [3]). In § 3, which deals, as the author states, with the combinations of natural numbers, two important operations on the combinations have been defined: the sum (*Summe*) and the intersection (*Durchschnitt*). The meaning of these terms is the same as nowadays. To denote sum, Dedekind uses +, while − is used to denote intersection.

The importance of operations is first illustrated by means of examples (cf. [3], p. 109).

So for combinations

$$\alpha = 2347,$$
$$\beta = 1357,$$
$$\gamma = 1267,$$

Sums and intersections are the same

$$\beta + \gamma = 123567, \qquad \alpha + \gamma = 123467, \qquad \alpha + \beta = 123457,$$

$$\beta - \gamma = 17, \qquad \gamma - \alpha = 27, \qquad \alpha - \beta = 37.$$

Then, a series of identities concerning combinatorics is presented

$$\alpha + \beta = \beta + \alpha,$$

$$\alpha - \beta = \beta - \alpha,$$

$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma),$$
$$(\alpha-\beta)-\gamma=\alpha-(\beta-\gamma),$$
$$\alpha+(\alpha-\beta)=\alpha,$$
$$\alpha-(\alpha+\beta)=\alpha,$$

He notices also that there exists a distribute law of subtraction $+$ in relation to $-$ and the other way round

$$(\alpha-\beta)+(\alpha-\gamma)=\alpha-(\beta+\gamma),$$

$$(\alpha+\beta)-(\alpha+\gamma)=\alpha+(\beta-\gamma).$$

In the next part of his article, the author extends operations $+$ and $-$ to more general structures of algebra: modules, fields, Abelian groups and ideals. It is probably this algebraic perspective that makes Dedekind ignore a difference of sets; an operation which lacks regular properties, as it is neither commutative nor associative. The concept of difference of sets was present in Boole's algebra and Cantor's analytic considerations. It can be thus assumed that the source of this notion lies in scientific research conducted in the 19^{th} century within logic and mathematical analysis.

References

[1] Dauben J. W., *Georg Cantor. His Mathematics and Philosophy*, Princeton University Press, Princeton, 1990

[2] Dedekind R., *Was sind und was sollen die Zahlen?*, siebente unveränderte Auflage, Friedrich Vieweg und Sohn, Braunschweig, 1939

[3] Dedekind R., *Gesammelte mathematische Werke*, Bd. II, Braunschweig, 1932

[4] Ferreirós J., *Labyrinth of Thought. A History of Set Theory and Its Role in Modern Mathematics*, Birkhäuser Verlag, Basel-Boston-Berlin, 1999

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