CREATIVE MATH.
14 (2005), 117 - 120

About generalization in mathematics (Part 1)

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ABSTRACT. In this paper we present the importance of generalization in the didactics of Mathematics with an illustration starting from a problem given at a mathematics competition.

1. GENERALIZATION IN MATHEMATICS AND DIDACTIC

Generalization is a logical operation on notions. We will adopt two of its senses [2]:

a) From a psychological point of view, it is on operation of thinking very connected with abstraction through and which common qualities of a group of objects or phenomena are mentally extended upon all the objects that belong to the category.

b) From a logical point of view, the operation opposed determination which consists moving from notions with a restricted area and with a rich content to notion with a larger area and a poorer content.

Actually, the first sense determines the second one and, together, the evolution of knowledge and development.

For example, we can talk about the well-known Pitagora's theorem.

First, it was generalized to all the triangles from the plan. Pitagora's theorem was known in the VIth century b. Ch., and its generalization in the plan was formulated by Papus from Alexandria in the IIIrd Century a. Ch.

The extension in the 3- dimensional space was done only in 1619 by R. Descartes.

Pitagora's pupils found three integer numbers so that:

$$x^2 + y^2 = z^2$$

After that, mathematicians searched for integer numbers so that:

$$x^3 + y^3 = z^3,$$

but they could not find any.

P. Fermat (1601-1665) said that "there are no natural numbers x, y, z that satisfy the relation:

$$x^n + y^n = z^n, \quad n > 2$$

known as "Fermat's Big Theorem" or Fermat's last theorem"

"I have discovered a truly wonderful demonstration but I don't have enough space to write it" noted the mathematician P. Fermat on the image of a page from the edition Diofant's Arithmetic, in 1637.

Received: 02.02.2004. In revised form: 12.04.2004
2000 *Mathematics Subject Classification.* 97B20, 05A05.
Key words and phrases. Generalization, group of permutations.

During a period of 358 of years, the world's great mathematicians tried, in vain, to find the demonstration.

The theory of divisibility of integer numbers was created in connection with Fermat's theorem. The theory of numbers is one of the most beautiful creation of the XIXth century.

The fundamental ideas of this theory are essential for the modern algebra.

In 1995, Andrew Wiles solved Fermat's last theorem in two articles containing 130 pages, published in Annals of Mathematics. His demonstration establishes deep connections between mathematics' domains which seem distant.

In didactic is, also, important to have in view the formation of notions and logical operations with notions. The main operations with notions are: determination, generalization, definition, division and classification.

Among those who had in view the formation of these mathematical operations in children, we mention the collective, who realized "Harward's Project for learning mathematics" presented by Z. P. Dienes in [1].

They started from the most simple sense of generalization, so to discover if x belongs to the class A, then it also belongs to class B. We say that A is included in B or class A can be extended to B. So, a linear space with a dimension is included in one with two dimensions. This could be the primary generalization. The more complex type of generalizations consists of establishing an isomorphism between a class and a subclass of another class.

A frequent form of generalization is the extension of the class of cases in which a certain thing can be done, transforming it from a finite class into an infinite class. When children learn that they can do a certain operation with some values of a variable, they realize the variable's value is irrelevant and that the operation can be done no matter which is the value.

Another type of generalization is moving from an infinite class to an infinite class, which is either called the primary generalization if the larger class includes the other class, either the mathematical generalization (as a matter) when it has an isomorph subclass but not identic to the prime class.

2. A generalization of a problem

At the Mathematics Olympiad in Austria the following problem was proposed

Solve the system:

$$\begin{cases} x = \dfrac{2y^2}{1+z^2} \\ y = \dfrac{2z^2}{1+x^2} \\ z = \dfrac{2x^2}{1+y^2}, \quad x, y, z \in \mathbb{R} \end{cases}$$

This problem does not require special knowledge in mathematics, but spirit of observation and good technique of calculating.

The solution to the problem is the following: we can see that $(0,0,0)$ is a solution. Indeed, if $x = 0$ then $y = 0$ and $z = 0$.

To obtain other solutions we can proceed in the following way: we multiply member with member and we obtain

$$(1+x^2)(1+y^2)(1+z^2) \geq 2^3 xyz$$

From the inequality

$$1+x^2 \geq 2x,\ \forall x \in \mathbb{R},$$

for $x=1$ we have the equality, we obtain the solution $(1,1,1)$.

There are no more solutions.

We have generalized this problem in the following way.

Let S_n be the group of permutations of the set

$$X = \{1, 2, ..., n\}$$

and

$$\sigma \in S_n.$$

Let us solve the system

$$(S_\sigma)\ \ X_i = \frac{2X_{\sigma(i)}^2}{1+X_{\sigma^{-1}(i)}^2},\ i = \overline{1,n}.$$

When σ is circular permutation, we obtain the simple generalization of the initial system and the solving is the same.

Let us solve, now, the system (S_σ). We notice that $(0,0,\ldots,0)$ is a solution. Also for every $i,\ i \in \{1,2,\ldots,n\},\ x_i \geq 0$.

We will prove that it allows $(1,1,\ldots,1)$ as a solution.

Indeed, if $x_k \neq 0,\ (\forall k),\ k=\overline{1,n}$ then $x_k > 0$. By multiplying the equations side by side we obtain:

$$\left(1+x_1^2\right)\left(1+x_2^2\right)\ldots\left(1+x_n^2\right) \geq 2^n x_1 x_2 \ldots x_n$$

and equality if and only if $x_k = 1,\ k = \overline{1,n}$.

Suppose that i is the first number so that $x_i = 0$, therefore it results that $x_{\sigma(i)} = 0$ and $x_{\sigma(\sigma(i))} = 0 \ldots$ until j for which $(\sigma \circ \sigma \circ \ldots \circ \sigma)(i) = i$

Let it be $\sigma(i) = i_1, \sigma(\sigma(i)) = i_2 \ldots$. We have $x_k = 0,\ k \in \{i_1, i_2, \ldots, i_j\}$. Hence we obtain

$$\sigma^{-1}(i_1) = i,\ \sigma^{-1}(i_2) = i_1, \ldots, \sigma^{-1}(i_j) = i_{j^{-1}}.$$

So

$$x_{\sigma^{-1}(k)} = 0,\ k \in \{i_1, \ldots, i_j\}.$$

If $x_k \neq 0,\ k \in \{i_1, i_2, \ldots, i_j\}$ then, like in the above case, $x_k = 1$.

From here, results that the system allows solutions like:

$$(1,1,\ldots,1,0,0,\ldots,0,1,\ldots)$$

such that every cycle of the permutation σ can have the values 0 or 1.

In conclusion, if $C(\sigma)$ is the number of the cycles of the permutation σ then the number of solutions of the system (S_σ) is $2^{C(\sigma)}$ (including the cycles with the length 1).

We obtain the maximum number of different solutions for the identic permutation, because it has n cycles with the length 1, so 2^n solutions.

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