CREATIVE MATH.
14 (2005), 121 - 126

The method of characteristic values in testing mathematical models

István Huba Atilla Sass

Abstract. Mathematical modeling of natural phenomena is frequently used by scientists. Since mathematical models usually involve laborious computation, there is a real need for shortcut methods testing the suitability of the chosen model in an early stage of the modeling. In this paper we present and illustrate such a method, namely the method of characteristic values.

1. Introduction

When we construct a mathematical model in order to study a given natural phenomenon or a given object in nature, we have to decide which physical laws we will use. In the construction of our mathematical model we have to consider not only universal lows but also certain very specific physical laws that characterize only the given natural phenomenon. Choosing these specific physical laws represents the key task in the construction of the mathematical model. To be sure that we made a right choice, we may perform a test.

It is well known that equations of physics must satisfy the dimension analysis, i.e. the dimensions on the left side of an equation must be exactly the same as on the right side of the equation.

Let we consider such an ecuation

$$F\left(X_1, X_2, \cdots, X_n\right) = G\left(X_1, X_2, \cdots, X_n\right) \tag{1}$$

where $X_i,\ i \in \overline{1,n}$ are quantities expressed in physical units (m, kg, s etc.). Denoting

$$X_i = a_i \cdot x_i, \ \ i \in \overline{1,n} \tag{2}$$

where a_i are constants having exactly the same dimensions as have X_i and x_i are the corresponding dimensionless quantities, from equation (1) results an equation of the form

$$A \cdot f\left(x_1, x_2, \cdots, x_n\right) = B \cdot g\left(x_1, x_2, \cdots, x_n\right) \tag{3}$$

where A and B are constants of the same dimensions. Taking

$$\frac{A}{B} = 1 \tag{4}$$

we deduce the simplified equation

$$f = g$$

Received: 11.10.2005. In revised form: 12.11.2005
2000 *Mathematics Subject Classification.* 85A15.
Key words and phrases. *Mathematical modeling, superdense star, white dwarf, neutron star*.

in the dimesionless quantities x_i.

The dimensionless quantities x_i will be of an order approximately equal with 1, so the computation related to the previous equation will be much easier. Performing similar simplifications as (4) for all equations that appear in our mathematical model, we obtain a system of equations of the form (4) having as unknowns the constants a_i, $i \in \overline{1,n}$.

We will call the solution $a_i^0, i \in \overline{1,n}$ of this system the *characteristic values* of the physical quantities X_i.

The order of characteristic values thus obtained must be close to the order of the values of the quantities X_i used in the mathematical model. Otherwise we can be sure that in our model we have considered some laws that are not suitable for the given natural phenomenon, and therefore we have to change our model. By changing the model at this moment we can avoid a large amount of useless computational work.

In case that our model seems to be suitable, we have to solve the system of simplified equations in dimensionless unknowns x_i and then from the solution of this system x_i^0, $i \in \overline{1,n}$ we obtain the physical solution

$$X_i^0 = a_i^0 x_i^0 \qquad i \in \overline{1,n} \tag{5}$$

that characterize the given phenomenon.

As a concrete example now we present a mathematical model of a superdense star.

2. Mathematical modeling of a superdense star

2.1. **Choosing the equations.** For building super dense, spherical and without rotation star models a system of differential equations is used
a) the state equations:

$$P = Af(x) \tag{6}$$

and

$$\rho = Bx^3 \tag{7}$$

where

$$f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3\ln[x + (x^2 + 1)^{1/2}] \tag{8}$$

x is the degeneration parameter of an electronic or neutron gas [1], [2]. For a degenerate electronic gas

$$A = \frac{\pi m_e^4 c^5}{3h^3} \tag{9}$$

$$B = \frac{8\pi m_e^3 c^3 H}{3h^3}\,\mu_e \tag{10}$$

and for a degenerate neutron gas

$$A = \frac{m_n^4 c^5}{24\pi^2 \hbar^3} \tag{11}$$

$$B = \frac{m_n^4 c^3}{3\pi^2 \hbar^3} \tag{12}$$

b) the relativistic hydrostatic equilibrium equation

$$\frac{dP(r)}{dr} = -G\left(\rho + \frac{P}{c^2}\right)\frac{M(r) + \frac{4\pi}{c^2}P\,r^3}{r^2\left(1 - \frac{2G}{c^2}\frac{M(r)}{r}\right)} \tag{13}$$

and

c) the equation of mass conservation

$$\frac{dM(r)}{dr} = 4\pi\rho r^2 \tag{14}$$

P and ρ represents the pressure and the density of stellar gas at the radius r, $M(r)$ is the mass of the star inner the radius r and

$c = 2.99792458 \cdot 10^8$ ms^{-1} – velocity of light;

$h = 2\pi\hbar = 6.6260755 \cdot 10^{-34}$ kgm^{2}s^{-1} – Planck's constant;

$G = 6.6732 \cdot 10^{-11}$ m^{3}kg^{-1}s^{-2} universal gravitational constant;

$m_e = 9.1093897 \cdot 10^{-31}$ kg – electron rest mass;

$H = 1.6726231 \cdot 10^{-27}$ kg – proton rest mass;

$m_n = 1.6749286 \cdot 10^{-27}$ - neutron rest mass;

μ_e - average molecular mass per free electron = 2 for helium

For a star with spherical symmetry we have the boundary conditions:
at centre

$$r = 0 :\ x = x_0,\ \rho = \rho_c\,,\ P = P_c\,,\ M(0) = 0$$

on the surface

$$r = R :\ x = 0\,,\ \rho = 0\,,\ P = 0\,,\ M(R) = M$$

In a white dwarf star the material is degenerate electronic gas in a neutron star degenerate neutron gas.

2.2. **Reduction to dimensionless magnitudes.** In order to facilitate the solving of the system of equations (6), (7), (8), (13), (14) we can introduce the dimensionless magnitudes m, η and χ, defined by the formulae:

$$M(r) = M_0 m\,,\ r = r_0\eta\,,\ \chi = (x^2 + 1)^{1/2} \tag{15}$$

where M_0 and r_0 represent respectively **the characteristic mass** and **the characteristic radius** expressed in kilograms and, respectively, in meters.

The equation (14) is transformed in

$$\frac{dm}{d\eta} = Dx^3\eta^2 \tag{16}$$

where the expression of D is:

$$D = 4\pi B\frac{r_0^3}{M_0}.$$

Taking $D = 1$, we get

$$M_0 = 4\pi B r_0^3 \tag{17}$$

If we introduce the following notations:

$$E = \frac{A}{Bc^2}\ , \quad F = \frac{4\pi GB}{c^2}r_o^2 \tag{18}$$

from (13), (8), (6) and (15) results

$$\frac{df}{d\eta} = -\frac{F}{E}\frac{(x^3+Ef)(m+Ef\eta^3)}{\eta^2\left(1-\frac{2Fm}{\eta}\right)} \tag{19}$$

$$\frac{df}{d\eta} = 8x^3\frac{d\chi}{d\eta} \tag{20}$$

and from (19) and (20) result the equation:

$$\frac{d\chi}{d\eta} = -\frac{F}{8E}\frac{(x^3+Ef)(m+Ef\eta^3)}{x^3\eta^2\left(1-\frac{Fm}{\eta}\right)} \tag{21}$$

If we choose

$$\frac{F}{8E} = 1 \tag{22}$$

we get the dimensionless system

$$\frac{dm}{d\eta} = x^3\eta^2 \tag{23}$$

$$\frac{d\chi}{d\eta} = -\frac{(x^3+Ef)(m+Ef\eta^3)}{x^3\eta^2\left(1-\frac{16Em}{\eta}\right)} \tag{24}$$

$$x = (\chi^2-1)^{1/2} \tag{25}$$

$$f = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3\ln[x + (x^2 + 1)^{1/2}] \tag{26}$$

From (18) and (22) results for the characteristic radius r_0

$$r_0 = \left(\frac{2A}{\pi G B^2}\right)^{1/2} \tag{27}$$

For the characteristic density ρ_0 we have

$$\rho_0 = \frac{3M_0}{4\pi r_0^3} = 3B \tag{28}$$

2.3. **Applications.**

a) For a **white dwarf star** from relations (27), (17), (28), (9) and (10) results

$r_0 = 3.88303 \cdot 10^6\text{m} = 0.60948 R_\oplus$

$M_0 = 1.43422 \cdot 10^{30}\text{kg} = 0.72108\, M_\odot$

$\rho_0 = 5.848101 \cdot 10^9 \text{kgm}^{-3}$

$R_\oplus$ is the terrestrial radius and $M_\odot$ the solar mass.
The characteristic density of a white dwarf star is $\approx 6.000.000\ g/cm^3$.

b) For a **neutron star (pulsar)** from relations (27), (17), (28), (11) and (12) we have

$r_0 = 4.191142 \cdot 10^3\text{m} = 4.191142$ km

$M_0 = 5.64633 \cdot 10^{30}\text{kg} = 2.83873\, M_\odot$

$\rho_0 = 1.83033 \cdot 10^{19}\text{kg/m}^3$

These values concords with the real values for a white dwarf star and, respectively, for a neutron star (pulsar).

The order of dimensionless variables η, f, χ, x, m is ~ 1 and the system (23)-(26) is handy.

References

[1] Landau L., Lifchitz E., *Physique théorique. Physique statistique*, Editura Mir, Moscova, 1967

[2] Sass I.H.A., *Late stages in stellar evolution*, Monographical booklets in applied & computer mathematics, MN-24/Pamm, Budapest University of Technology and Economics, Budapest, 2002

University of Baia Mare
Department of Mathematics
Victoriei 76, 430122 Baia Mare
Romania
E-mail address: huba@ubm.ro