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Semi-inscribed circles in a triangle

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ABSTRACT. This article analyzes circles with the following property: two sides of a given triangle are tangent to the circle and the circle has a joint point with the circle circumscribed to the triangle. We are going to call this kind of circles **semi-inscribed** circles. Every triangle has three semi-inscribed circles. For this particular type of circles we will establish new properties that you have not found yet in the mathematical literature.

1. ELEMENTS OF TRIANGLE GEOMETRY

For ABC triangle we are going to use the standard notations.

The following triangle properties are known:

1.1. $\cos(\angle OAI) = \cos\frac{\beta-\gamma}{2}$, $\cos(\angle OBI) = \cos\frac{\gamma-\alpha}{2}$, $\cos(\angle OCI) = \cos\frac{\alpha-\beta}{2}$
(I is the centre of the ABC triangle's inscribed circle, O is the centre of the ABC triangle's circumscribed circle).

Proof.

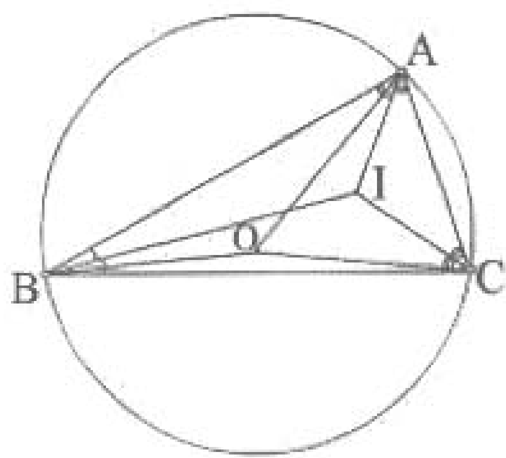

Fig. 1

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by the property 1.1.

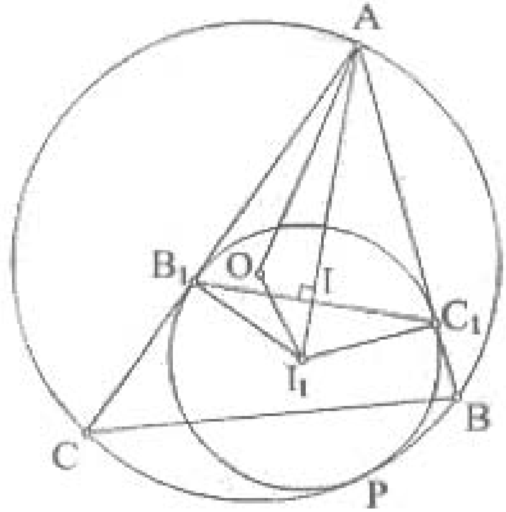

Fig. 2

Next, we note with P the joint point of those two circles, the circumscribed circle and the semi-inscribed circle. It is clear that points O, I_1 and P are collinear. Now, in ABC triangle we apply the cosine rule. After equivalent changes we obtain the following relation: $4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=r_1\cos^2\frac{\alpha}{2}$.

Using this formula and the property 1.2. we get to the statement.

If r_2 is the radius of the semi-inscribed circle for which BA and BC sides are tangent and r_3 is the radius of the third semi-inscribed circle then

$$r_2=\frac{r}{\cos^2\frac{\beta}{2}},\quad r_3=\frac{r}{\cos^2\frac{\gamma}{2}}.$$

3. New properties for the radii of the ABC triangle semi-inscribed circles

3.1. Properties obtained using the Viète relations for the third degree equation. It is known that the equation

$$4R^2x^3-4R(R+r)x^2+\left(p^2+r^2-4R^2\right)x+(2R+r)^2-p^2=0 \qquad (3.1)$$

has the solutions $\cos\alpha,\cos\beta$ and $\cos\gamma$ (α,β and γ are the measures of the ABC triangle angles) [2].

In the first identity making the substitution $x=2y-1$ (taking into account that $\cos\alpha=2\cos^2\frac{\alpha}{2}-1$) we obtain the equation

$$16R^2y^3-8R(4R+r)y^2+\left(p^2+(4R+r)^2\right)y-p^2=0 \qquad (3.2)$$

$$r_1r_2 + r_2r_3 + r_3r_1 = \frac{8Rr^2(4R+r)}{p^2} \tag{3.13}$$

$$r_1r_2r_3 = \frac{16R^2r^3}{p^2} \tag{3.14}$$

3.2. **Identities that can be created applying the symmetric polynomial and symmetric functions properties.** If we have $P_1 = x + y + z$, $P_2 = xy + yz + zx$ and $P_3 = xyz$ then the following equalities are true:

$$\frac{x+y}{xy} + \frac{y+z}{yz} + \frac{z+x}{zx} = \frac{2P_2}{P_3} \tag{3.15}$$

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \left(\frac{P_2}{P_3}\right)^2 - \frac{2P_1}{P_3} \tag{3.16}$$

$$\frac{x+y}{z} \cdot \frac{y+z}{x} \cdot \frac{z+x}{y} = \frac{P_1P_2}{P_3} - 1 \tag{3.17}$$

$$\frac{x+y}{z} + \frac{y+z}{x} + \frac{z+x}{y} = \frac{P_1P_2}{P_3} - 3 \tag{3.18}$$

From identities (3.15), (3.16), (3.17) and (3.18) we obtain some new relations:

$$\frac{r_1+r_2}{r_1r_2} + \frac{r_2+r_3}{r_2r_3} + \frac{r_3+r_1}{r_3r_1} = \frac{4R+r}{Rr} \tag{3.19}$$

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{(4R+r)^2 - p^2}{8R^2r^2} \tag{3.20}$$

$$\frac{r_1+r_2}{r_3} \cdot \frac{r_2+r_3}{r_1} \cdot \frac{r_3+r_1}{r_2} = \frac{1}{2}\left(1 + \left(\frac{4R+r}{p}\right)^2\right)\left(4 + \frac{r}{R}\right) - 1 \tag{3.21}$$

$$\frac{r_1+r_2}{r_3} \cdot \frac{r_2+r_3}{r_1} \cdot \frac{r_3+r_1}{r_2} - 2 = \frac{r_1+r_2}{r_3} + \frac{r_2+r_3}{r_1} + \frac{r_3+r_1}{r_2} \tag{3.22}$$