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New refinements for the AM-GM-HM inequalities

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ABSTRACT. In this paper we introduce new means which give new refinements for the AM-GM-HM inequalities.

1. MAIN RESULTS

Theorem 1. *If* $x_k > 0\ (k = 1, 2, ..., n)$, *then*

$$\frac{1}{n}\sum_{k=1}^{n} x_k \geq \frac{1}{2n}\sum_{cyclic}\sqrt[3]{x_1x_2}\left(\sqrt[3]{x_1}+\sqrt[3]{x_2}\right) \geq \sqrt[n]{\prod_{k=1}^{n} x_k} \geq \frac{2n}{\sum\limits_{cyclic}\frac{1}{\sqrt[3]{x_1x_2}}\left(\frac{1}{\sqrt[3]{x_1}}+\frac{1}{\sqrt[3]{x_2}}\right)} \geq \frac{n}{\sum\limits_{k=1}^{n}\frac{1}{x_k}}$$

Proof. $a_1^3 + a_2^3 = (a_1 + a_2)\left(a_1^2 + a_2^2 - a_1a_2\right) \geq a_1a_2(a_1 + a_2)$,

$$2\sum_{k=1}^{n} a_k^3 = \sum_{cyclic}\left(a_1^3 + a_2^3\right) \geq \sum_{cyclic} a_1a_2(a_1 + a_2) \geq 2n\sqrt[n]{\prod_{k=1}^{n} a_k^3}.$$

If $a_k = \sqrt[3]{x_k}\ (k = 1, 2, ..., n)$, then

$$\frac{1}{n}\sum_{k=1}^{n} x_k \geq \frac{1}{2n}\sum_{cyclic}\sqrt[3]{x_1x_2}\left(\sqrt[3]{x_1}+\sqrt[3]{x_2}\right) \geq \sqrt[n]{\prod_{k=1}^{n} x_k}.$$

In this we take $x_k \to \frac{1}{x_k}\ (k = 1, 2, ..., n)$, therefore

$$\sqrt[n]{\prod_{k=1}^{n} x_k} \geq \frac{2n}{\sum\limits_{cyclic}\frac{1}{\sqrt[3]{x_1x_2}}\left(\frac{1}{\sqrt[3]{x_1}}+\frac{1}{\sqrt[3]{x_2}}\right)} \geq \frac{n}{\sum\limits_{k=1}^{n}\frac{1}{x_k}}.$$

Remark. $B(x_1, x_2, ..., x_n) = \frac{1}{2n}\sum\limits_{cyclic}\sqrt[3]{x_1x_2}\left(\sqrt[3]{x_1}+\sqrt[3]{x_2}\right)$ and

$$\overline{B}(x_1, x_2, ..., x_n) = \frac{2n}{\sum\limits_{cyclic}\frac{1}{\sqrt[3]{x_1x_2}}\left(\frac{1}{\sqrt[3]{x_1}}+\frac{1}{\sqrt[3]{x_2}}\right)}$$

are new means introduced in [1].

The inequalities of theorem can be written $A \geq B \geq G \geq \overline{B} \geq H$.

Application 1.1. In all triangles ABC holds:

1) $2\left(s^2 - 3r^2 - 6Rr\right) \geq s^2 + r^2 - 2Rr \geq 12Rr$

2) $s\left(s^2 - 12Rr\right) \geq \frac{r}{2}(4R + r) - sr^2 \geq 3sr^2$

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3) $(4R+r)^3-12s^2R\ge s^2(2R-r)\ge 3s^2r$

4) $(2R-r)\left((4R+r)^2-3s^2\right)+6Rr^2\ge$

$$\ge\frac{1}{2}(2R-r)\left(s^2+r^2-8Rr\right)-3Rr^2\ge 6Rr^2$$

5) $(4R+r)^3-3s^2(2R+r)\ge\frac{1}{2}\left((4R+r)^3+s^2r\right)-s^2R\ge 6s^2R$

Proof. In inequalities $\sum x^3\ge\frac{1}{2}\prod(x+y)-xyz\ge 6xyz$, we take $(x,y,z)\in\left\{(a,b,c),(s-a,s-b,s-c),(r_a,r_b,r_c);\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$. These inequalities are refinements for classical triangle inequalities (Euler, Gerretsen etc.).

Application 1.2. In all tetrahedrons $ABCD$ holds:

1)

$$\frac{1}{4}\sum h_a\ge\frac{1}{8}\sum\sqrt[3]{h_ah_b}\left(\sqrt[3]{h_a}+\sqrt[3]{h_b}\right)\ge\sqrt[4]{\prod h_a}\ge\frac{8}{\sum\frac{1}{\sqrt[3]{h_ah_b}}\left(\frac{1}{\sqrt[3]{h_a}}+\frac{1}{\sqrt[3]{h_b}}\right)}\ge 4r$$

2)

$$\frac{1}{4}\sum r_a\ge\frac{1}{8}\sum\sqrt[3]{r_ar_b}\left(\sqrt[3]{r_a}+\sqrt[3]{r_b}\right)\ge\sqrt[4]{\prod r_a}\ge\frac{8}{\sum\frac{1}{\sqrt[3]{r_ar_b}}\left(\frac{1}{\sqrt[3]{r_a}}+\frac{1}{\sqrt[3]{r_b}}\right)}\ge 2r$$

Proof. In Theorem 1 we take $n=4$ and

$$(x,y,z,t)\in\{(h_a,h_b,h_c,h_d);(r_a,r_b,r_c,r_d)\}.$$

Application 1.3. In all triangles ABC holds:

1)

$$\frac{2s}{3}\ge\frac{1}{6}\sum\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\ge\sqrt[3]{4sRr}\ge\frac{6}{\sum\frac{1}{\sqrt[3]{ab}}\left(\frac{1}{\sqrt[3]{a}}+\frac{1}{\sqrt[3]{b}}\right)}\ge\frac{12sRr}{s^2+r^2+4Rr}$$

2)

$$\frac{s}{3}\ge\frac{1}{6}\sum\sqrt[3]{(s-a)(s-b)}\left(\sqrt[3]{s-a}+\sqrt[3]{s-b}\right)\ge\sqrt[3]{sr^2}\ge$$

$$\ge\frac{6}{\sum\frac{1}{\sqrt[3]{(s-a)(s-b)}}\left(\frac{1}{\sqrt[3]{s-a}}+\frac{1}{\sqrt[3]{s-b}}\right)}\ge\frac{3sr}{4R+r}$$

3)

$$\frac{4R+r}{3}\ge\frac{1}{6}\sum\sqrt[3]{r_ar_b}\left(\sqrt[3]{r_a}+\sqrt[3]{r_b}\right)\ge\sqrt[3]{s^2r}\ge\frac{6}{\sum\frac{1}{\sqrt[3]{r_ar_b}}\left(\frac{1}{\sqrt[3]{r_a}}+\frac{1}{\sqrt[3]{r_b}}\right)}\ge 3r$$

4)

$$\frac{s}{3r}\ge\frac{1}{6}\sum\sqrt[3]{ctg\frac{A}{2}ctg\frac{B}{2}}\left(\sqrt[3]{ctg\frac{A}{2}}+\sqrt[3]{ctg\frac{B}{2}}\right)\ge\sqrt[3]{\frac{s}{r}}\ge$$

$$\ge\frac{6}{\sum\sqrt[3]{tg\frac{A}{2}tg\frac{B}{2}}\left(\sqrt[3]{tg\frac{A}{2}}+\sqrt[3]{tg\frac{B}{2}}\right)}\ge\frac{3s}{4R+r}$$

5)

$$\frac{4R+r}{3s} \geq \frac{1}{6}\sum \sqrt[3]{tg\frac{A}{2}tg\frac{B}{2}}\left(\sqrt[3]{tg\frac{A}{2}}+\sqrt[3]{tg\frac{B}{2}}\right) \geq \sqrt[3]{\frac{r}{s}} \geq$$

$$\geq \frac{6}{\sum \sqrt[3]{ctg\frac{A}{2}ctg\frac{B}{2}}\left(\sqrt[3]{ctg\frac{A}{2}}+\sqrt[3]{ctg\frac{B}{2}}\right)} \geq \frac{3r}{s}$$

6)

$$\frac{2R-r}{6R} \geq \frac{1}{6}\sum \sqrt[3]{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\left(\sqrt[3]{\sin^2\frac{A}{2}}+\sqrt[3]{\sin^2\frac{B}{2}}\right) \geq \sqrt[3]{\frac{r^2}{16R^2}} \geq$$

$$\geq \frac{6}{\sum \frac{1}{\sqrt[3]{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}}\left(\frac{1}{\sqrt[3]{\sin^2\frac{A}{2}}}+\frac{1}{\sqrt[3]{\sin^2\frac{B}{2}}}\right)} \geq \frac{3r^2}{s^2+r^2-8Rr}$$

7)

$$\frac{4R+r}{6R} \geq \frac{1}{6}\sum \sqrt[3]{\cos^2\frac{A}{2}\cos^2\frac{B}{2}}\left(\sqrt[3]{\cos^2\frac{A}{2}}+\sqrt[3]{\cos^2\frac{B}{2}}\right) \geq \sqrt[3]{\frac{s^2}{16R^2}} \geq$$

$$\geq \frac{6}{\sum \frac{1}{\sqrt[3]{\cos^2\frac{A}{2}\cos^2\frac{B}{2}}}\left(\frac{1}{\sqrt[3]{\cos^2\frac{A}{2}}}+\frac{1}{\sqrt[3]{\cos^2\frac{B}{2}}}\right)} \geq \frac{3s^2}{s^2+(4R+r)^2}$$

Proof. In Theorem 1 we take $n=3$ and

$$(x,y,z) \in \{(a,b,c),(s-a,s-b,s-c),(r_a,r_b,r_c);$$

$$\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right);$$

$$\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\Bigg\}.$$

These inequalities give new refinements for classical triangle inequalities (Euler, Gerretsen etc.).

Application 1.4. We have the following inequalities:

$$1+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} \geq B\left(2,\frac{2}{3},...,\frac{n+1}{n}\right) \geq \sqrt[n]{n+1} \geq$$

$$\geq \overline{B}\left(2,\frac{3}{2},...,\frac{n+1}{m}\right) \geq \frac{1}{1-\frac{1}{n}\sum\limits_{k=1}^{n}\frac{1}{k+1}}$$

Application 1.5. We have the following inequalities:

$$1-\frac{1}{n}\sum_{k=2}^{n+1}\frac{1}{k} \geq B\left(\frac{1}{2},\frac{2}{3},...,\frac{n}{n+1}\right) \geq \frac{1}{\sqrt[n]{n+1}} \geq$$

$$\geq \overline{B}\left(\frac{1}{2},\frac{2}{3},...,\frac{n}{n+1}\right)\geq \frac{1}{1+\frac{1}{n}\sum\limits_{k=2}^{n+1}\frac{1}{k-1}}$$

Application 1.6. We have the following inequalities:

$$1-\frac{1}{n}\sum_{k=2}^{n+1}\frac{1}{k^2}\geq B\left(\frac{2^2-1}{2^2},\frac{3^2-1}{3^2},...,\frac{(n+1)^2-1}{(n+1)^2}\right)\geq \sqrt[n]{\frac{n+1}{2(n+1)}}\geq$$

$$\geq \overline{B}\left(\frac{2^2-1}{2^2},\frac{3^2-1}{3^2},...,\frac{(n+1)^2-1}{(n+1)^2}\right)\geq \frac{1}{1+\frac{1}{n}\sum\limits_{k=2}^{n+1}\frac{1}{k^2-1}}$$

Application 1.7. We have the following inequalities:

$$\frac{n}{n+1}\geq B\left(\frac{1}{1\cdot 2},\frac{1}{2\cdot 3},...,\frac{1}{n(n+1)}\right)\geq \frac{1}{\sqrt[n]{(n+1)(n!)^2}}\geq$$

$$\geq \overline{B}\left(\frac{1}{1\cdot 2},\frac{1}{2\cdot 3},...,\frac{1}{n(n+1)}\right)\geq \frac{3}{(n+1)(n+2)}$$

Application 1.8. We have the following inequalities:

$$\frac{2^n-1}{n}\geq B\left(1,2,2^2,...,2^{n-1}\right)\geq 2^{\frac{n-1}{2}}\geq$$

$$\geq \overline{B}\left(1,2,2^2,...,2^{n-1}\right)\geq \frac{n2^{n-1}}{2^n-1}$$

Theorem 2. *If* $x_i>0\ (i=1,2,...,n)$, $\alpha\in(-\infty,0]\cup[1,+\infty)$, $k\in\{1,2,...,n\}$, *then*

$$\frac{1}{n}\sum_{i=1}^{n}x_i^{\alpha}\geq \frac{1}{n}\sum_{cyclic}\left(\frac{x_1+...+x_k}{k}\right)^{\alpha}\geq\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^{\alpha}\geq\left(\sqrt[n]{\prod_{i=1}^{n}x_i}\right)^{\alpha}\geq$$

$$\geq\left(\frac{n}{\sum\limits_{i=1}^{n}\frac{1}{x_i}}\right)^{\alpha}\geq \frac{n}{\sum\limits_{cyclic}\left(\frac{\frac{1}{x_1}+...+\frac{1}{x_k}}{k}\right)^{\alpha}}\geq \frac{n}{\sum\limits_{i=1}^{n}\frac{1}{x_i^{\alpha}}}.$$

Proof.

$$\frac{1}{n}\sum_{i=1}^{n}x_i^{\alpha}=\frac{1}{n}\sum_{cyclic}\frac{x_1^{\alpha}+...+x_k^{\alpha}}{k}\geq\frac{1}{n}\sum_{cyclic}\left(\frac{x_1+...+x_k}{k}\right)^{\alpha}\geq$$

$$\geq\left(\frac{1}{n}\sum_{cyclic}\frac{x_1+...+x_k}{k}\right)^{\alpha}=\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^{\alpha}\geq\left(\sqrt[n]{\prod_{i=1}^{n}x_i}\right)^{\alpha}.$$

If in this we take $x_i\to\frac{1}{x_i}\ (i=1,2,...,n)$, then we obtain the another inequalities.

Application 2.1. In all triangles ABC holds

1)

$$\frac{2\left(s^2-r^2-4Rr\right)}{3} \geq \frac{1}{6}\left(3s^2-r^2-4Rr\right) \geq \left(\frac{2s}{3}\right)^2 \geq \left(\sqrt[3]{4sRr}\right)^2 \geq$$

$$\geq 9\left(\frac{4sRr}{s^2+r^2+4Rr}\right)^2 \geq \frac{12}{\sum\left(\frac{1}{a}+\frac{1}{b}\right)^2} \geq \frac{3}{\left(\frac{s^2+r^2+4Rr}{4sRr}\right)^2-\frac{1}{Rr}}$$

2)

$$\frac{s^2-2r^2-8Rr}{3} \geq \frac{s^2-r^2-4Rr}{6} \geq \left(\frac{s}{3}\right)^2 \geq \left(\sqrt[3]{sr^2}\right) \geq$$

$$\geq 9\left(\frac{sr}{4R+r}\right)^2 \geq \left(\frac{12}{\sum\left(\frac{c}{(s-a)(s-b)}\right)^2}\right) \geq \frac{3s^2r^2}{(4R+r)^2-2s^2}$$

3)

$$\frac{(4R+r)^2-2s^2}{3} \geq \frac{(4R+r)^2-s^2}{6} \geq \left(\frac{4R+r}{3}\right)^2 \geq \left(\sqrt[3]{s^2r}\right)^2 \geq$$

$$\geq 9r^2 \geq \frac{12}{\sum\left(\frac{1}{r_a}+\frac{1}{r_b}\right)^2} \geq \frac{3s^2r^2}{s^2-8Rr-2r^2}$$

Proof. In Theorem 2 we take $n=3$, $\alpha=2$ so we obtain

$$\frac{x^2+y^2+z^2}{3} \geq \frac{(x+y)^2+(y+z)^2+(z+x)^2}{12} \geq \left(\frac{x+y+z}{3}\right)^2 \geq (\sqrt[3]{xyz})^2 \geq$$

$$\geq \frac{9}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2} \geq \frac{12}{\left(\frac{1}{x}+\frac{1}{y}\right)^2+\left(\frac{1}{y}+\frac{1}{z}\right)^2+\left(\frac{1}{z}+\frac{1}{x}\right)^2} \geq \frac{3}{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}$$

and in this we take

$$(x,y,z) \in \left\{(a,b,c),(s-a,s-b,s-c),(r_a,r_b,r_c)\right\}.$$

Theorem 3. *If* $\alpha \in (-\infty,0] \cup [1,+\infty)$ *and* $x_i>0$ $(i=1,2,\ldots,n)$, *then*

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \frac{1}{n}\sum_{cyclic}\left(\frac{x_1^{\frac{1}{\alpha}}+\ldots+x_k^{\frac{1}{\alpha}}}{k}\right)^{\alpha} \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\frac{1}{\alpha}}\right)^{\alpha} \geq \sqrt[n]{\prod_{i=1}^{n} x_i} \geq$$

$$\geq \left(\frac{n}{\sum\limits_{i=1}^{n} x_i^{-\frac{1}{\alpha}}}\right)^{\alpha} \geq \frac{n}{\sum\limits_{cyclic}\left(\frac{x_1^{-\frac{1}{\alpha}}+\ldots+x_k^{-\frac{1}{\alpha}}}{k}\right)^{\alpha}} \geq \frac{n}{\sum\limits_{i=1}^{n}\frac{1}{x_i}}.$$

Proof. In Theorem 2 we take $x_i \to x_i^{\frac{1}{\alpha}}$ $(i=1,2,\ldots,n)$. These inequalities are new refinements for AM-GM-HM inequalities.

Remark. The following

$$C_\alpha(x_1,x_2,\ldots,x_n) = \frac{1}{n}\sum_{cyclic}\left(\frac{x_1^{\frac{1}{\alpha}}+\ldots+x_k^{\frac{1}{\alpha}}}{k}\right)^{\alpha},$$

$$D_\alpha(x_1, x_2, ..., x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\frac{1}{\alpha}}\right)^\alpha,$$

$$\overline{D_\alpha}(x_1, x_2, ..., x_n) = \left(\frac{n}{\sum\limits_{i=1}^{n} x_i^{-\frac{1}{\alpha}}}\right)^\alpha,$$

$$\overline{C_\alpha}(x_1, x_2, ..., x_n) = \frac{n}{\sum\limits_{cyclic}\left(\frac{x_1^{-\frac{1}{\alpha}}+...+x_k^{-\frac{1}{\alpha}}}{k}\right)^\alpha}$$

are new means for which $A \geq C_\alpha \geq D_\alpha \geq G \geq \overline{D_\alpha} \geq \overline{C_\alpha} \geq H$. These where introduced in [1].

Application 3.1. In all triangles ABC holds

1)

$$\frac{2s}{3} \geq C_\alpha(a, b, c) \geq D_\alpha(a, b, c) \geq \sqrt[3]{4sRr} \geq$$
$$\geq \overline{D_\alpha}(a, b, c) \geq \overline{C_\alpha}(a, b, c) \geq \frac{12sRr}{s^2 + r^2 + 4Rr}$$

2)

$$\frac{s}{3} \geq C_\alpha(s-a, s-b, s-c) \geq D_\alpha(s-a, s-b, s-c) \geq \sqrt[3]{sr^2} \geq$$
$$\geq \overline{D_\alpha}(s-a, s-b, s-c) \geq \overline{C_\alpha}(s-a, s-b, s-c) \geq \frac{3sr}{4R+r}$$

3)

$$\frac{4R+r}{3} \geq C_\alpha(r_a, r_b, r_c) \geq D_\alpha(r_a, r_b, r_c) \geq \sqrt[3]{s^2 r} \geq$$
$$\geq \overline{D_\alpha}(r_a, r_b, r_c) \geq \overline{C_\alpha}(r_a, r_b, r_c) \geq 3r$$

4)

$$\frac{2R-r}{6R} \geq C_\alpha\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right) \geq D_\alpha\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right) \geq \sqrt[3]{\frac{r^2}{16R^2}} \geq$$
$$\geq \overline{D_\alpha}\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right) \geq \overline{C_\alpha}\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right) \geq \frac{3r^2}{s^2 + r^2 - 8Rr}$$

5)

$$\frac{4R+r}{6R} \geq C_\alpha\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right) \geq D_\alpha\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right) \geq \sqrt[3]{\frac{s^2}{16R^2}} \geq$$
$$\geq \overline{D_\alpha}\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right) \geq \overline{C_\alpha}\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right) \geq \frac{3s^2}{s^2 + (4R+r)^2}$$

These inequalities are new refinements for a lot of classical triangle inequalities (Euler, Gerretsen etc.).

Theorem 4. *If $f : R \to R$ is convex and $x_i \in R$ $(i = 1, 2, ..., n)$, $k \in \{1, 2, ..., n\}$, then*

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq \frac{1}{n}\sum_{cyclic} f\left(\frac{x_1 + x_2 + ... + x_k}{k}\right) \geq f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

If f is concave then we have the reverse inequalities. This is a refinement of Jensen's inequality.

Proof.

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq \frac{1}{n}\sum_{cyclic} \frac{f(x_1) + ... + f(x_k)}{k} \geq \frac{1}{n}\sum_{cyclic} f\left(\frac{x_1 + ... + x_k}{k}\right) \geq$$

$$\geq f\left(\frac{1}{n}\sum_{cyclic} \frac{x_1 + ... + x_k}{k}\right) = f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Application 4.1. If $x_i > 0$ $(i = 1, 2, ..., n)$, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^n \geq \prod_{cyclic} \frac{x_1 + x_2 + ... + x_k}{k} \geq \prod_{i=1}^{n} x_i \geq \frac{1}{\prod\limits_{cyclic} \frac{\frac{1}{x_1} + ... + \frac{1}{x_k}}{k}} \geq \left(\frac{n}{\sum\limits_{i=1}^{n} \frac{1}{x_i}}\right)^n$$

a new refinement of AM-GM-HM inequalities.

Proof. In Theorem 4 we take $f(x) = \ln x$.

Theorem 5. *If $f : R \to R$ is bijective and convex, $g : R \to R$ is bijective and concave $x_i \in R$ $(i = 1, 2, ..., n)$, $k \in \{1, 2, ..., n\}$, then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} f^{-1}(x_i)\right) \leq \frac{1}{n}\sum_{cyclic} f\left(\frac{f^{-1}(x_1) + ... + f^{-1}(x_k)}{k}\right) \leq$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} x_i \leq \frac{1}{n}\sum_{cyclic} f\left(\frac{g^{-1}(x_1) + ... + g^{-1}(x_k)}{k}\right) \leq g\left(\frac{1}{n}\sum_{i=1}^{n} g^{-1}(x_i)\right).$$

Proof. The function f is convex and bijective, therefore in Theorem 4 we take $x_i \to f^{-1}(x_i)$ $(i = 1, 2, ..., n)$, so we obtain

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \frac{1}{n}\sum_{cyclic} f\left(\frac{f^{-1}(x_1) + ... + f^{-1}(x_k)}{k}\right) \geq f\left(\frac{1}{n}\sum_{i=1}^{n} f^{-1}(x_i)\right).$$

The function g is concave and bijective, therefore in Theorem 4 we take $x_i \to g^{-1}(x_i)$ $(i = 1, 2, ..., n)$, so we obtain

$$\frac{1}{n}\sum_{i=1}^{n} x_i \leq \frac{1}{n}\sum_{cyclic} g\left(\frac{g^{-1}(x_1) + ... + g^{-1}(x_k)}{k}\right) \leq g\left(\frac{1}{n}\sum_{i=1}^{n} g^{-1}(x_i)\right).$$

Application 5.1. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$\frac{1}{\ln\left(\frac{1}{n}\sum\limits_{i=1}^{n} e^{\frac{1}{x_i}}\right)} \leq \frac{n}{\sum\limits_{cyclic} \ln\left(\frac{e^{\frac{1}{x_1}} + ... + e^{\frac{1}{x_k}}}{k}\right)} \leq \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{x_i}} \leq \frac{n}{\sum\limits_{cyclic} \frac{1}{\sqrt[k]{x_1 ... x_k}}} \leq$$

$$\le \sqrt[n]{\prod_{i=1}^{n} x_i} \le \frac{1}{n}\sum_{cyclic} \sqrt[k]{x_1 ... x_k} \le \frac{1}{n}\sum_{i=1}^{n} x_i \le \frac{1}{n}\sum_{cyclic} \ln\left(\frac{e^{x_1}+...+e^{x_k}}{k}\right) \le \ln\left(\frac{1}{n}\sum_{i=1}^{n} e^{x_i}\right).$$

These are new refinements for AM-GM-GM inequalities.

Proof. In Theorem 5 we take $f(x) = e^x$ and $g(x) = \ln x$.

Remark. The following

$$\begin{aligned} F_1(x_1, x_2, ..., x_n) &= \frac{1}{\ln\left(\frac{1}{n}\sum\limits_{i=1}^{n} e^{\frac{1}{x_i}}\right)}, \\ F_2(x_1, x_2, ..., x_n) &= \frac{n}{\sum\limits_{cyclic} \ln\left(\frac{e^{\frac{1}{x_1}}+...+e^{\frac{1}{x_k}}}{k}\right)}, \\ F_3(x_1, x_2, ..., x_n) &= \frac{n}{\sum\limits_{cyclic} \frac{1}{\sqrt[k]{x_1 x_2 ... x_k}}}, \\ \overline{F_t}(x_1, x_2, ..., x_n) &= \frac{1}{F_t(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n})}, (t \in \{1,2,3\}) \end{aligned}$$

are new means for which $F_1 \le F_2 \le H \le F_3 \le G \le \overline{F_3} \le A \le \overline{F_2} \le \overline{F_1}$.
If $x_i \in [m, M]$ $(i = 1, 2, ..., n)$, then $m \le F_1 \le \overline{F_1} \le M$.

Application 5.2. In all triangles ABC holds:

1)

$$F_1(a,b,c) \le F_2(a,b,c) \le \frac{12sRr}{s^2+r^2+4Rr} \le$$
$$\le F_3(a,b,c) \le \sqrt[3]{4sRr} \le \overline{F_3}(a,b,c) \le \frac{2s}{3} \le \overline{F_2}(a,b,c) \le \overline{F_1}(a,b,c).$$

2)

$$F_1(s-a,s-b,s-c) \le F_2(s-a,s-b,s-c) \le \frac{3sr}{4R+r} \le F_3(s-a,s-b,s-c) \le \sqrt[3]{sr^2} \le$$
$$\le \overline{F_3}(s-a,s-b,s-c) \le \frac{s}{3} \le \overline{F_2}(s-a,s-b,s-c) \le \overline{F_1}(s-a,s-b,s-c).$$

3)

$$F_1(r_a,r_b,r_c) \le F_2(r_a,r_b,r_c) \le 3r \le F_3(r_a,r_b,r_c) \le \sqrt[3]{s^2 r} \le$$
$$\le \overline{F_3}(r_a,r_b,r_c) \le \frac{4R+r}{3} \le \overline{F_2}(r_a,r_b,r_c) \le \overline{F_1}(r_a,r_b,r_c).$$

Theorem 6. *If $f : R \to R$ is convex, bijective and increasing, $g : R \to R$ is concave, bijective and increasing, $x_i \in R$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then*

$$g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right) \le g^{-1}\left(\frac{1}{n}\sum_{cyclic} f\left(\frac{x_1+...+x_k}{k}\right)\right) \le \frac{1}{n}\sum_{i=1}^{n} x_i \le$$
$$\le f^{-1}\left(\frac{1}{n}\sum_{cyclic} f\left(\frac{x_1+...+x_k}{k}\right)\right) \le f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right).$$

If f and g are decreasing, then we have the reverse inequalities.

Application 6.1. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$\frac{1}{\ln\left(\frac{1}{n}\sum_{i=1}^{n} e^{\frac{1}{x_i}}\right)} \leq \frac{1}{\ln\left(\frac{1}{n}\sum_{cyclic} e^{\frac{x_1+...+x_k}{k}}\right)} \leq \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} \leq \frac{1}{\sqrt[n]{\prod_{cyclic}\frac{\frac{1}{x_1}+...+\frac{1}{x_k}}{k}}} \leq$$

$$\leq \sqrt[n]{\prod_{i=1}^{n} x_i} \leq \sqrt[n]{\prod_{cyclic}\frac{x_1+...+x_k}{k}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i \leq \ln\left(\frac{1}{n}\sum_{cyclic} e^{\frac{x_1+...+x_k}{k}}\right) \leq \ln\left(\frac{1}{n}\sum_{i=1}^{n} e^{x_i}\right).$$

These are new refinements for AM-GM-HM inequalities.

Proof. In Theorem 5 we take $f(x) = e^x$and $g(x) = \ln x$.

Remark. The following

$$\begin{aligned} F_4(x_1, x_2, ..., x_n) &= \frac{1}{\ln\left(\frac{1}{n}\sum_{cyclic} e^{\frac{x_1+...+x_k}{k}}\right)}, \\ F_5(x_1, x_2, ..., x_n) &= \frac{1}{\sqrt[n]{\prod_{cyclic}\frac{\frac{1}{x_1}+...+\frac{1}{x_k}}{k}}}, \\ \overline{F_t}(x_1, x_2, ..., x_n) &= \frac{1}{F_t(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n})}, (t \in \{4, 5\}) \end{aligned}$$

are new means for which $F_1 \leq F_4 \leq H \leq F_5 \leq G \leq \overline{F_5} \leq A \leq \overline{F_4} \leq \overline{F_1}$.

Application 6.2. In all triangles ABC holds:

1)

$$F_1(a, b, c) \leq F_4(a, b, c) \leq \frac{12sRr}{s^2 + r^2 + 4Rr} \leq F_5(a, b, c) \leq \sqrt[3]{4sRr} \leq$$

$$\leq \overline{F_3}(a, b, c) \leq \frac{2s}{3} \leq \overline{F_4}(a, b, c) \leq \overline{F_1}(a, b, c).$$

2)

$$F_2(s-a, s-b, s-c) \leq F_2(s-a, s-b, s-c) \leq \frac{3sr}{4R + r} \leq F_5(s-a, s-b, s-c) \leq \sqrt[3]{sr^2} \leq$$

$$\leq \overline{F_5}(s-a, s-b, s-c) \leq \frac{s}{3} \leq \overline{F_4}(s-a, s-b, s-c) \leq \overline{F_1}(s-a, s-b, s-c).$$

3)

$$F_2(r_a, r_b, r_c) \leq F_4(r_a, r_b, r_c) \leq 3r \leq F_5(r_a, r_b, r_c) \leq \sqrt[3]{s^2 r} \leq$$

$$\leq \overline{F_5}(r_a, r_b, r_c) \leq \frac{4R + r}{3} \leq \overline{F_4}(r_a, r_b, r_c) \leq \overline{F_1}(r_a, r_b, r_c).$$

References

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