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# About the coefficients of Bernstein multivariate polynomial

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ABSTRACT. In this note we want to determinate the coefficients of Bernstein polynomial associated to the functions $e_p(x) = x^p$, $p \in \mathbb{N}$, connected with the Stirling numbers of second kind.

## 1. PRELIMINARIES

Let $B_m : C[0,1] \to C[0,1]$, $m$ a non zero natural number, be the Bernstein operators defined for any function $f \in C[0,1]$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials, defined by

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \quad k \in \{0, 1, \dots, m\} \tag{1.2}$$

and $x \in [0,1]$.

For $x \in \mathbb{R}$, $k \in \mathbb{N}$, let $x^{[k]} = x(x-1)\dots(x-k+1)$, $x^{[0]} = 1$. It is well known that

$$x^k = \sum_{\nu=1}^{k} S(k,\nu) x^{[\nu]}, \quad x \in \mathbb{R}, \quad k \in \mathbb{N}^*, \tag{1.3}$$

where $S(k,\nu)$, $\nu \in \{1, 2, \dots, k\}$ are the Stirling numbers of second kind. These numbers verify the relations

$$\begin{aligned} &S(p,k) = kS(p-1,k) + S(p-1,k-1),\ S(1,1) = 1, \\ &S(2,1) = S(2,2) = 1,\ S(p,1) = S(p,p) = 1, \end{aligned} \tag{1.4}$$

for $p \in \mathbb{N}$, $p \geq 3$, $k \in \{2, 3, \dots, p-1\}$ and then can be calculated with (1.4)

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 3 & & 1 & & & \\
 & & 1 & & 7 & & 6 & & 1 & & \\
 & 1 & & 15 & & 25 & & 10 & & 1 & \\
1 & & 31 & & 90 & & 65 & & 15 & & 1 \\
 & & & & \dots & \dots & \dots & \dots & & &
\end{array}$$

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We note $S(p,k)=0$ from definition if $p,k\in\mathbb{N}$, $p<k$.

In the $k$-dimensional case, we have

$$(B_nf)(x_1,\dots,x_k)=\sum_{\nu_i\geq 0,\,\nu_1+\dots+\nu_k\leq n} p_{\nu_1,\dots,\nu_k;n}(x_1,\dots,x_k)f\left(\frac{\nu_1}{n},\dots,\frac{\nu_k}{n}\right), \quad (1.5)$$

$$\begin{aligned} &p_{\nu_1,\dots,\nu_k;n}(x_1,\dots,x_k)=\\ &=\binom{n}{\nu_1,\dots,\nu_k}x_1^{\nu_1}\dots x_k^{\nu_k}(1-x_1-\dots-x_k)^{n-\nu_1-\dots-\nu_k}, \quad \text{with} \end{aligned}$$

$$\binom{n}{\nu_1,\dots,\nu_k}=\frac{n!}{\nu_1!\dots\nu_k!(n-\nu_1-\dots-\nu_k)!},$$

where $f$ and $p_{\nu_1,\dots,\nu_k;n}$ are defined on $k$-dimensional simplex

$$\Delta_k: x_i\geq 0,\ i=1,\dots,k,\ x_1+\dots+x_k\leq 1 \quad (1.6)$$

and $f\in C(\Delta_k)$.

## 2. Main results

In paper [2] we proved that

$$(B_me_p)=\frac{1}{m^p}\sum_{k=1}^{p}m^{[k]}S(p,k)x^k,\ m,p\in\mathbb{N}^*. \quad (2.1)$$

In the following, we want to prove an analogous relations for the Bernstein multivariate polynomial.

For the Bernstein bivariate polynomial we have

**Proposition 1.** *If $m,p,q\in\mathbb{N}^*$, then*

$$(B_me_{pq})(x,y)=\frac{1}{m^{p+q}}\sum_{i=1}^{p}\sum_{j=1}^{q}m^{[i+j]}S(p,i)S(q,j)x^iy^j, \quad (2.2)$$

*where $e_{pq}(x,y)=x^py^q$, $x,y\in\Delta_2$.*

*Proof.* We have

$$\begin{aligned}(B_me_{pq})(x,y)&=\sum_{k,l=0,k+l\leq m}\frac{m!}{k!\,l!(m-k-l)!}x^ky^l(1-x-y)^{m-k-l}\frac{k^p}{m^p}\frac{l^q}{m^q}=\\ &=\frac{1}{m^{p+q}}\sum_{k,l=0,k+l\leq m}\frac{m!}{k!\,l!(m-k-l)!}x^ky^l(1-x-y)^{m-k-l}k^pl^q.\end{aligned}$$

But $k^p=\sum_{i=1}^{p}S(p,i)k^{[i]}$ and $l^q=\sum_{j=1}^{q}S(q,j)l^{[j]}$ so that

$$k^pl^q=\sum_{i=1}^{p}\sum_{j=1}^{q}S(p,i)S(q,j)k^{[i]}l^{[j]}.$$

Now, we have

$$(B_m e_{pq})(x,y) = \frac{1}{m^{p+q}} \sum_{i=1}^{p} \sum_{j=1}^{q} S(p,i)S(q,j) \sum_{k,l=0,k+l\le m} \frac{m!}{k!\,l!(m-k-l)!} \cdot$$

$$\cdot\, x^k y^l (1-x-y)^{m-k-l} k^{[i]} l^{[j]} =$$

$$= \frac{1}{m^{p+q}} \sum_{i=1}^{p} \sum_{j=1}^{q} S(p,i)S(q,j) x^i y^j \cdot$$

$$\cdot \sum_{k=i,l=j,k+l\le m} \frac{(m-i-j)!m^{[i+j]}}{(k-i)!(l-j)!(m-k-l)!}\, x^{k-i} y^{l-j} (1-x-y)^{m-k-l} =$$

$$= \frac{1}{m^{p+q}} \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} S(p,i)S(q,j) x^i y^j$$

because

$$\sum_{k=i,l=j,k+l\le m} \frac{(m-i-j)!}{(k-i)!(l-j)!(m-k-l)!}\, x^{k-i} y^{l-j} (1-x-y)^{m-k-l} =$$

$$= (x+y+1-x-y)^{m-i-j} = 1.$$

□

In a similar way, we can prove

**Proposition 2.** *If* $m, p_1, \ldots, p_k \in \mathbb{N}^*$, $k \in \mathbb{N}$, $k \ge 3$, *then*

$$\left(B_m e_{p_1 \ldots p_k}\right)(x_1, \ldots, x_k) = \frac{1}{m^{p_1+\cdots+p_k}} \sum_{n_1=1}^{p_1} \cdots \tag{2.3}$$

$$\cdots \sum_{n_k=1}^{p_k} m^{[n_1+\cdots+n_k]} S(p_1,n_1) \ldots S(p_k,n_k) x_1^{n_1} \ldots x_k^{n_k},$$

*where* $e_{p_1 \ldots p_k}(x_1, \ldots, x_k) = x_1^{p_1} \ldots x_k^{p_k}$, $x_1, \ldots x_k \in \Delta_k$.

**Remark 2.1.** Obviously,

$$(B_m e_{p0})(x,y) = (B_m e_p)(x),\ m \ge p \tag{2.4}$$

and

$$(B_m e_{0q})(x,y) = (B_m e_q)(y),\ m \ge q. \tag{2.5}$$

**Application 2.1.** We have

$$(B_m e_{11})(x,y) = \frac{m-1}{m}\, xy \tag{2.6}$$

$$(B_m e_{21})(x,y) = \frac{(m-1)(m-2)}{m^2}\, x^2 y + \frac{m-1}{m^2}\, xy \tag{2.7}$$

$$(B_m e_{31})(x,y) = \frac{(m-1)(m-2)(m-3)}{m^3}\, x^3 y + \tag{2.8}$$

$$+ \frac{3(m-1)(m-2)}{m^3}\, x^2 y + \frac{m-1}{m^3}\, xy$$

$$(B_m e_{41})(x,y) = \frac{(m-1)(m-2)(m-3)(m-4)}{m^4}\,x^4y + \tag{2.9}$$
$$+\,\frac{6(m-1)(m-2)(m-3)}{m^4}\,x^3y +$$
$$+\,\frac{7(m-1)(m-2)}{m^4}\,x^2y + \frac{m-1}{m^4}\,xy$$

$$(B_m e_{22})(x,y) = \frac{(m-1)(m-2)(m-3)}{m^3}\,x^2y^2 + \tag{2.10}$$
$$+\,\frac{(m-1)(m-2)}{m^3}\,(x^2y + xy^2) + \frac{m-1}{m^3}\,xy$$

$$(B_m e_{32})(x,y) = \frac{(m-1)(m-2)(m-3)(m-4)}{m^4}\,x^3y^2 + \tag{2.11}$$
$$+\,\frac{(m-1)(m-2)(m-3)}{m^4}\,x^3y + \frac{3(m-1)(m-2)(m-3)}{m^4}\,x^2y^2 +$$
$$+\,\frac{3(m-1)(m-2)}{m^4}\,x^2y + \frac{(m-1)(m-2)}{m^4}\,xy^2 + \frac{m-1}{m^4}\,xy$$

$$(B_m e_{42})(x,y) = \frac{(m-1)(m-2)(m-3)(m-4)(m-5)}{m^5}\,x^4y^2 + \tag{2.12}$$
$$+\,\frac{(m-1)(m-2)(m-3)(m-4)}{m^5}\,x^4y +$$
$$+\,\frac{6(m-1)(m-2)(m-3)(m-4)}{m^5}\,x^3y^2 +$$
$$+\,\frac{6(m-1)(m-2)(m-3)}{m^5}\,x^3y +$$
$$+\,\frac{7(m-1)(m-2)(m-3)}{m^5}\,x^2y^2 +$$
$$+\,\frac{7(m-1)(m-2)}{m^5}x^2y + \frac{(m-1)(m-2)}{m^5}xy^2 + \frac{m-1}{m^5}xy.$$

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