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Method of steps for mixed second order functional-differential equations

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ABSTRACT. The main purpose of this paper is to apply the method of steps for mixed second order functional differential equations. The linear case is discussed as an example.

1. THE MAIN RESULT

In what follows we shall consider the problem:

$$x''(t) = f(t, x(t), x(t-h), x(t+h)), \quad t \in [a,b] \tag{1.1}$$
$$x(t) = \varphi(t), \quad t \in [t_0 - h, t_0 + h] \tag{1.2}$$

where $t_0 \in [a,b]$, $a \le t_0 - h, t_0 + h \le b$ and $\varphi \in C^2[t_0 - h, t_0 + h]$.

Let $n_a := \left[\dfrac{t_0 - a}{h}\right]$, $n_b := \left[\dfrac{b - t_0}{h}\right]$ and $n := \max\{n_a, n_b\}$.

By a solution of (1.1) + (1.2) we understand a function

$$x \in C[a-h, b+h] \cap C^2[a,b]$$

which satisfies (1.1) + (1.2) for all $t \in [a,b]$.

We consider the following conditions:

Let $f \in C^{n+2}([a,b] \times \mathbb{R}^3)$.

(C1) *For all* $u_1 \in [a,b], u_2, u_4, u_5 \in \mathbb{R}$, *there exist a unique* $u_3 \in \mathbb{R}$,

$$u_3 = f_1(u_1, u_2, u_4, u_5),\ f_1 \in C^{n+2}([a,b] \times \mathbb{R}^3)$$

such that $u_5 = f(u_1, u_2, u_3, u_4)$.

(C2) *For all* $u_1 \in [a,b], u_2, u_3, u_5 \in \mathbb{R}$, *there exist a unique* $u_4 \in \mathbb{R}$,

$$u_4 = f_2(u_1, u_2, u_3, u_5),\ f_2 \in C^{n+2}([a,b] \times \mathbb{R}^3)$$

such that $u_5 = f(u_1, u_2, u_3, u_4)$.

We have the result:

Theorem 1.1. *Let* $f \in C^{n+2}([a,b] \times \mathbb{R}^3)$ *which satisfies* $(C1)$ *and* $(C2)$.
If $\varphi \in C^{n+2}[t_0 - h, t_0 + h]$, *then the problem* $(1.1) + (1.2)$ *has a unique solution*

$$x \in C^n[a-h, b+h] \cap C^{n+2}[a,b].$$

If φ *satisfies the condition:*

$$\varphi^{(k+2)}(t_0) = \left[f(t, \varphi(t), \varphi(t-h), \varphi(t+h))\right]^{(k)}_{t=t_0},\ k \in \{0,1,2,...,n\}, \tag{1.3}$$

then $x \in C^{n+2}[a-h, b+h]$.

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Proof. By the method of steps we construct the solution of $(1.1) + (1.2)$ as follows:

Let $t \in [t_0, t_0 + h]$. Then $\varphi''(t) = f(t, \varphi(t), \varphi(t-h), x(t+h))$. From condition (C2) we have:

$$x(t) := x_1(t) = f_2(t-h, \varphi(t-h), \varphi(t-2h), \varphi''(t-h)), t \in [t_0 + h, t_0 + 2h].$$

By the same method we find the final step:

$$x_{n_b}(t) = f_2(t-h, x_{n_b-1}(t-h), x_{n_b-1}(t-2h), x''_{n_b-1}(t-h)), t \in [t_0 + n_b \cdot h, b],$$

where $n_b = \left[\frac{b-t_0}{h}\right]$. We must have

$$\varphi(t_0 + h) = x_1(t_0 + h)$$

$$x_p(t_0 + (p+1)h) = x_{p+1}(t_0 + (p+1)h), \quad p \leq n_b - 1$$

In the same way we have the solution on $[a, t_0]$ with the condition: $\varphi(t_0 - h) = x_{-1}(t_0 - h)$; $x_{-p}(t_0 - (p+1)h) = x_{-(p+1)}(t_0 - (p+1)h)$, $p \leq n_a - 1$, where $n_a = \left[\frac{t_0 - a}{h}\right]$. So, the solution is:

$$x(t) = \begin{cases} x_{-n_a}(t), & \text{if } t \in [a, t_0 - n_a h], \\ x_{-k}(t), & \text{if } t \in [t_0 - (k+1)h, t_0 - kh], \\ 1 \leq k \leq n_a - 1 \varphi(t), & \text{if } t \in [t_0 - h, t_0 + h], \\ x_k(t), & \text{if } t \in [t_0 + kh, t_0 + (k+1)h], \\ 1 \leq k \leq n_b - 1 x_{n_b}(t), & \text{if } t \in [t_0 + n_b h, b]. \end{cases}$$

Let $n := \max\{n_a, n_b\}$. Now we prove the necessity of the condition (1.3).

Let $x \in C[a-h, b+h] \cap C^2[a, b]$ be a solution of the problem (1.1) + (1.2). If $x \in C^n[a-h, b+h] \cap C^{n+2}[a, b]$, then:

$$x^{(k+2)}(t) = [f(t, x(t), x(t-h), x(t+h))]^{(k)} \text{ for all } t \in [a, b], k \in \{0, 1, 2, ..., n\}.$$

For $t = t_0$ we have:

$$\varphi^{(k+2)}(t_0) = \left[f(t, \varphi(t), \varphi(t-h), \varphi(t+h))\right]^{(k)}_{t=t_0}, \; k \in \{0, 1, 2, ..., n\}.$$

Since $\varphi \in C^{n+2}[t_0 - h, t_0 + h]$, we infer that $x \in C^{n+2}[a-h, b+h]$. □

2. Example

We consider the following example:

$$x''(t) = \alpha \cdot x(t) + \beta \cdot x(t-h) + \gamma \cdot x(t+h), t \in [a, b] \tag{2.4}$$

$$x(t) = \varphi(t), t \in [t_0 - h, t_0 + h], \tag{2.5}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $\beta \neq 0$, $\gamma \neq 0, t_0 \in [a, b]$ and $a \leq t_0 - h$; $t_0 + h \leq b$.

To find conditions for the existence of a solution of the problem (2.4) + (2.5) we apply the method of steps on intervals $[t_0, b]$ and $[a, t_0]$.

Let $t \in [t_0, t_0 + h]$,

$$\varphi''(t) = \alpha \cdot \varphi(t) + \beta \cdot \varphi(t-h) + \gamma \cdot \varphi(t+h).$$

Then

$$x(t) := x_1(t) = \frac{1}{\gamma}\left[\alpha \cdot \varphi(t-h) + \beta \cdot \varphi(t-2h) - \varphi''(t-h)\right],$$

for $t \in [t_0 + h, t_0 + 2h]$.

Let $t \in [t_0 + h, t_0 + 2h]$,

$$x_1''(t) = \alpha \cdot x_1(t) + \beta \cdot \varphi(t-h) + \gamma \cdot x(t+h).$$

Then

$$x(t) := x_2(t) = \frac{1}{\gamma} \cdot [\alpha \cdot x_1(t-h) + \beta \cdot \varphi(t-2h) - x_1''(t-h)],$$

for $t \in [t_0 + 2h, t_0 + 3h]$.

In the same way the final step on $[t_0, b]$ we obtain:

$$x_{n_b}(t) = \frac{1}{\gamma} \cdot [\alpha \cdot x_{n_b-1}(t-h) + \beta \cdot x_{n_b-2}(t-2h) - x_{n_b-1}''(t-h)],$$

for $t \in [t_0 + n_b \cdot h, b]$, where $n_b = \left[\frac{b-t_0}{h}\right]$.

On interval $[a, t_0]$ we find that:

$$x_{n_a}(t) = \frac{1}{\gamma} \cdot [\alpha \cdot x_{n_a-1}(t+h) + \beta \cdot x_{n_a-2}(t+2h) - x_{n_a-1}''(t+h)],$$

for $t \in [a, t_0 - n_a \cdot h]$, where $n_a = \left[\frac{t_0-a}{h}\right]$.

Let $n := \max\{n_a, n_b\}$, $\varphi \in C^{n+2}[t_0-h, t_0+h]$ and $x \in C^n[a-h, b+h] \cap C^{n+2}[a,b]$ be a solution of the problem (2.4) + (2.5).

We have:

$$x^{(k+2)}(t) = \alpha \cdot x^{(k)}(t) + \beta \cdot x^{(k)}(t-h) + \gamma \cdot x^{(k)}(t+h),$$

for $k \in \{0, 1, 2, ..., n\}$.

For $t = t_0$ we obtain:

$$\varphi^{(k+2)}(t_0) = \alpha \cdot \varphi^{(k)}(t_0) + \beta \cdot \varphi^{(k)}(t_0-h) + \gamma \cdot \varphi^{(k)}(t_0+h),$$

for $k \in \{0, 1, 2, ..., n\}$.

Then the problem (2.4) + (2.5) has a solution if and only if

$$\varphi^{(k+2)}(t_0) = \alpha \cdot \varphi^{(k)}(t_0) + \beta \cdot \varphi^{(k)}(t_0-h) + \gamma \cdot \varphi^{(k)}(t_0+h),$$

for $k \in \{0, 1, 2, ..., n\}$.

For example, if

$$\alpha = \beta = 1,\ \gamma = -1,\ h = \pi,\ \varphi(t) = \sin(\pi - t)$$
$$t_0 = 0,\ a = -2\pi,\ b = 2\pi,\ t \in [-\pi, \pi]$$

we have the following solution:

$$x(t) = \sin(\pi - t),\ t \in [-2\pi, 2\pi].$$

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