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Geometric prequantization of the dual of Lie algebra $so(6)$

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ABSTRACT. We consider the Lie algebra $so(6)$ of the special orthogonal Lie group $SO(6)$ and we give the construction of the Lie-Poisson structure on the dual of $so(6)$. The goal of our paper is to construct a geometric prequantization of the Poisson manifold $so(6)^*$, using the Weinstein theory of symplectic groupoids.

1. INTRODUCTION

In the last time was a great deal of interest in the study of Poisson manifolds and of Lie groupoids in connection with their deep applications in differential geometry, symplectic geometry and quantum mechanics; see, for instance Mackenzie [9], Mikami and Weinstein [11] and Puta [12]. Among the most important subjects of the theory of Poisson manifolds is of course the problem of their quantization from the geometric quantization point of view. Moreover, the theory of geometric quantization was extended to the Poisson manifolds via the theory of symplectic groupoids in the sense of Karasev and Weinstein.

The problem of geometric prequantization of a Poisson manifold was one of the principal motivations behind the introduction of symplectic groupoids in Karasev ([7], 1987) and Weinstein ([14] , 1987). There are some many results in this direction in a series of papers; see for details [8], [12] - [15].

In the papers of Gh. Ivan ([3]), Gh. Ivan and Popuţa ([4], [5]) are given the constructions of the geometric prequantization of the Poisson manifold $so(n)^*$, for $n = 3, 4$ and 5. In the sequel, we want to discuss this problem for the Poisson manifold $so(6)^*$.

2. THE LIE ALGEBRA $so(6)$

The Lie algebra $so(6)$ is the algebra of all skew-symmetric matrices of type 6×6 with real coefficients and the Lie bracket $[\cdot, \cdot]$ given by the commutator of matrix, i.e. $[A, B] = AB - BA$, for all $A, B \in so(6)$. More precisely, we have:

$$so(6) = \left\{ \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^T & A_{22} \end{pmatrix} | A_{11},\ A_{12},\ A_{22} \in M_3\,(R) \right\},$$

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where

$$A_{11}=\begin{pmatrix}0 & a_1 & a_2\\ -a_1 & 0 & a_6\\ -a_2 & -a_6 & 0\end{pmatrix},\quad A_{12}=\begin{pmatrix}a_3 & a_4 & a_5\\ a_7 & a_8 & a_9\\ a_{10} & a_{11} & a_{12}\end{pmatrix},$$

$$A_{22}=\begin{pmatrix}0 & a_{13} & a_{14}\\ -a_{13} & 0 & a_{15}\\ -a_{14} & -a_{15} & 0\end{pmatrix}$$

and $a_i \in \mathbf{R}$ for $i=\overline{1,15}$. We denote by $\varepsilon_{i,j}$ the matrix of type 6×6 having all elements equal with zero except the element of the position (i,j) which is equal to one. Then the set of skew-symmetric matrices $\{\varepsilon_{i,j}-\varepsilon_{j,i}|i<j, i,j=\overline{1,n}\}$ is a basis of the Lie algebra $so(6)$. Hence, $so(6)$ is a Lie algebra of dimension 15. Thus, if we take $\widehat{E}=\{E_i|i=\overline{1,15}\}$, where:

$$\begin{cases} E_i &= \varepsilon_{1,i+1}-\varepsilon_{i+1,1} \quad \text{for } i=\overline{1,5}\\ E_{5+j} &= \varepsilon_{2,j+2}-\varepsilon_{j+2,2} \quad \text{for } j=\overline{1,4}\\ E_{9+k} &= \varepsilon_{3,k+3}-\varepsilon_{k+3,3} \quad \text{for } k=\overline{1,3}\\ E_{12+s} &= \varepsilon_{4,s+4}-\varepsilon_{s+4,4} \quad \text{for } s=\overline{1,2}\\ E_{15} &= \varepsilon_{5,6}-\varepsilon_{6,5}\end{cases} \tag{1}$$

we obtain the canonical basis of the Lie algebra $so(6)$.

It follows that, for all $A\in so(6)$ there exists $\alpha^i\in\mathbf{R}$, $i=\overline{1,15}$ such that $A=\alpha^i E_i,\ i=\overline{1,15}$.. Also, for all $A,B\in so(6)$ we have $[A,B]=[\alpha^i E_i,\beta^j E_j]=\alpha^i\beta^j[E_i,E_j],\ i,j=\overline{1,15}$, where

$$[E_i,E_j] = c_{i,j}^k E_k,\ i,j,k=\overline{1,15}. \tag{2}$$

The real numbers $c_{i,j}^k, i,j,k=\overline{1,15}$ from (2) are called *the structure constants of the Lie algebra* $so(6)$.

Proposition 1. *The nonnulls structure constants of the Lie algebra $so(6)$ in the base $\widehat{E}$ are given in the following relations*:

$$\begin{cases} c_{1,i}^{i+4}=-1, i=\overline{2,5}; & c_{1,6}^2=-1; & c_{1,j}^{j-4}=1, j=\overline{7,9}; & c_{2,i}^{i+7}=-1, i=\overline{3,5}\\ c_{2,6}^1=-1; & c_{2,j}^{j-7}=1, j=\overline{10,12}; & c_{3,i}^{i+9}=-1, i=4,5; & c_{3,7}^1=-1\\ c_{3,10}^2=-1; & c_{3,j}^{j-9}=1, j=13,14; & c_{4,5}^{15}=-1; & c_{4,8}^1=-1\\ c_{4,11}^2=-1; & c_{4,13}^3=-1; & c_{4,15}^5=1; & c_{5,9}^1=-1\\ c_{5,12}^2=-1; & c_{5,14}^3=-1; & c_{5,15}^4=-1 & \end{cases} \tag{3.1}$$

$$\begin{cases} c_{6,7}^{10}=-1; & c_{6,8}^{11}=-1; & c_{6,9}^{12}=-1; & c_{6,10}^7=1; & c_{6,11}^8=1; & c_{6,12}^9=1\\ c_{7,8}^{13}=-1; & c_{7,9}^{14}=-1; & c_{7,10}^6=-1; & c_{7,13}^8=1; & c_{7,14}^9=1; & c_{8,9}^{15}=-1\\ c_{8,11}^6=-1; & c_{8,13}^7=-1; & c_{8,15}^9=1; & c_{9,12}^6=-1; & c_{9,14}^7=-1; & c_{9,15}^8=-1\\ c_{10,11}^{13}=-1; & c_{10,12}^{14}=-1; & c_{10,13}^{11}=1; & c_{10,14}^{12}=1; & c_{11,12}^{15}=-1; & c_{11,13}^{10}=-1\\ c_{12,14}^{10}=-1; & c_{12,15}^{11}=-1; & c_{13,14}^{15}=-1; & c_{13,15}^{14}=1; & c_{14,15}^{13}=-1 & \end{cases} \tag{3.2}$$

$$c_{ij}^k=-c_{ji}^k \quad for\ values\ of,\ i\ and\ j\ in\ (3.1)\ and\ (3.2). \tag{3.3}$$

Proof. Applying the properties of Lie brackets, we observe that $[E_i, E_i] = 0,$ for all $i = \overline{1, 15}$ and $[E_i, E_j] = -[E_j, E_i],$ for all $i, j = \overline{1, 15}, i \neq j.$ It suffices to calculate the Lie brackets $[E_i, E_j]$ for all $i, j = \overline{1, 15}$ such that $i < j.$

Using the relation $\varepsilon_{i,j} \cdot \varepsilon_{s,t} = \begin{cases} 0 & j \neq s \\ \varepsilon_{i,t} & j = s \end{cases}$ we obtain the brackets $[E_i, E_j],\ i < j,\ i, j = \overline{1, 15}$ expressed in the basis $\{E_i \mid i = \overline{1, 15}\}$ as in the following tables:

$[\cdot,\cdot]$	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
E_1	0	$-E_6$	$-E_7$	$-E_8$	$-E_9$	$-E_2$	E_3	E_4	E_5
E_2	E_6	0	$-E_{10}$	$-E_{11}$	$-E_{12}$	$-E_1$	0	0	0
E_3	E_7	E_{10}	0	$-E_{13}$	$-E_{14}$	0	$-E_1$	0	0
E_4	E_8	E_{11}	E_{13}	0	$-E_{15}$	0	0	$-E_1$	0
E_5	E_9	E_{12}	E_{14}	E_{15}	0	0	0	0	$-E_1$
E_6	E_2	E_1	0	0	0	0	$-E_{10}$	$-E_{11}$	$-E_{12}$
E_7	$-E_3$	0	E_1	0	0	E_{10}	0	$-E_{13}$	$-E_{14}$
E_8	$-E_4$	0	0	E_1	0	E_{11}	E_{13}	0	$-E_{15}$
E_9	$-E_5$	0	0	0	E_1	E_{12}	E_{14}	E_{15}	0

$[\cdot,\cdot]$	E_{10}	E_{11}	E_{12}	E_{13}	E_{14}	E_{15}
E_1	0	0	0	0	0	0
E_2	E_3	E_4	E_5	0	0	0
E_3	$-E_2$	0	0	E_4	E_5	0
E_4	0	$-E_2$	0	$-E_3$	0	E_5
E_5	0	0	$-E_2$	0	$-E_3$	$-E_4$
E_6	E_7	E_8	E_9	0	0	0
E_7	$-E_6$	0	0	E_8	E_9	0
E_8	0	$-E_6$	0	$-E_7$	0	E_9
E_9	0	0	$-E_6$	0	$-E_7$	$-E_8$

$[\cdot,\cdot]$	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
E_{10}	0	$-E_3$	E_2	0	0	$-E_7$	E_6	0	0
E_{11}	0	$-E_4$	0	E_2	0	$-E_8$	0	E_6	0
E_{12}	0	$-E_5$	0	0	E_2	$-E_9$	0	0	E_6
E_{13}	0	0	$-E_4$	E_3	0	0	$-E_8$	E_7	0
E_{14}	0	0	$-E_5$	0	E_3	0	$-E_9$	0	E_7
E_{15}	0	0	0	$-E_5$	E_4	0	0	$-E_9$	E_8

$[\cdot,\cdot]$	E_{10}	E_{11}	E_{12}	E_{13}	E_{14}	E_{15}
E_{10}	0	$-E_{13}$	$-E_{14}$	E_{11}	E_{12}	0
E_{11}	E_{13}	0	$-E_{15}$	$-E_{10}$	0	E_{12}
E_{12}	E_{14}	E_{15}	0	0	$-E_{10}$	$-E_{11}$
E_{13}	$-E_{11}$	E_{10}	0	0	$-E_{15}$	E_{14}
E_{14}	$-E_{12}$	0	E_{10}	E_{15}	0	$-E_{13}$
E_{15}	0	$-E_{12}$	E_{11}	$-E_{14}$	E_{13}	0

.

From equality $[E_1, E_2] = -E_6 = \sum_{i=1}^{15} c_{1,2}^{i} E_i$, we obtain the structure constants $c_{1,2}^{6} = -1$ and $c_{1,2}^{j} = 0$, for $j = \overline{1,15}, j \neq 6$.

In the same manner, we obtain successively the structure constants of the Lie algebra $so(6)$, given by the relations $(3.1) - (3.3)$. □

Let $\widehat{e} = \{e_i \mid i = \overline{1,15}\ \}$ be the canonical basis of the vector space $\mathbf{R}^{15}$. Consider the bijection $f : \widehat{e} = \{\ e_i \mid i = \overline{1,15}\ \} \longrightarrow \widehat{E} = \{\ E_i \mid i = \overline{1,15}\ \}$ defined by $f(e_i) = E_i, (\forall)\ i = \overline{1,15}$, whith $E_i \in so(5)$ given by (1) and satisfying the condition $f(-e_i) = -E_i,\ (\forall)\ i = \overline{1,15}$.

We define on $\widehat{e}$ the brackets $[e_i, e_j],\ i, j = \overline{1,15}$ by:

$$[e_i, e_j] = -[e_j, e_i] = f^{-1}([f(e_i), f(e_j)]), \text{ for } i < j \text{ and } [e_i, e_i] = 0. \tag{4}$$

For example, $[e_1, e_2] = f^{-1}([f(e_1), f(e_2)]) = f^{-1}([E_1, E_2]) = f^{-1}(-E_6) = -e_6$.

It is clearly that :

$$[e_i, e_j] = c_{i,j}^{k} e_k,\ i, j, k = \overline{1,15} \tag{5}$$

where $c_{i,j}^{k}$ are given in Proposition 1 by the relations (3.1)- (3.3).

If we define now the bracket on $\mathbf{R}^{15}$ by:

$$[x, y] = c_{i,j}^{k} x^i y^j e_k,\ i, j, k = \overline{1,15} \tag{6}$$

where $x = x^i e_i, y = y^j e_j$ and $c_{i,j}^{k}$ are given in $(3.1) - (3.3)$, then $(\mathbf{R}^{15}, +, \cdot, [\cdot, \cdot])$ is a Lie algebra over $\mathbf{R}$ of dimension 15.

We denote by $\widehat{f} : \mathbf{R}^{15} \longrightarrow so(6)$ the extension by linearity of the bijection f, i.e.

$$\widehat{f} : x = (x^1, x^2, \ldots, x^{15}) \in \mathbf{R}^{15} \longrightarrow \widehat{f}(x) = \widehat{x} \in so(6) \tag{7}$$

where $\widehat{x} = \begin{pmatrix} X_{11} & X_{12} \\ -X_{12}^T & X_{22} \end{pmatrix}$ with $X_{11}, X_{12}, X_{22} \in \mathcal{M}_3(\mathbf{R})$ given by

$X_{11} = \begin{pmatrix} 0 & x^1 & x^2 \\ -x^1 & 0 & x^6 \\ -x^2 & -x^6 & 0 \end{pmatrix}$ $X_{12} = \begin{pmatrix} x^3 & x^4 & x^5 \\ x^7 & x^8 & x^9 \\ x^{10} & x^{11} & x^{12} \end{pmatrix}$ and

$X_{22} = \begin{pmatrix} 0 & x^{13} & x^{14} \\ -x^{13} & 0 & x^{15} \\ -x^{14} & -x^{15} & 0 \end{pmatrix}$

We can prove the following proposition.

Proposition 2. *The map $\widehat{f}$ is an isomorphism of the Lie algebra $\mathbf{R}^{15}$ onto the Lie algebra $so(6)$.*

3. THE LIE-POISSON STRUCTURE ON THE DUAL OF LIE ALGEBRA $so(6)$

Let P be a smooth m-dimensional manifold and $C^\infty(P, \mathbf{R})$ the algebra of functions of C^∞- class from P to $\mathbf{R}$. A *Poisson manifold* is a pair $(P, \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$ is a bilinear operation on $C^\infty(P, \mathbf{R})$ such that $(C^\infty(P, \mathbf{R})$ is a Lie algebra and $\{\cdot, \cdot\}$ satisfies the Leibniz identity in each argument; that is, for all $a, b \in \mathbf{R}$ and $f, g, h \in C^\infty(P, \mathbf{R})$, the map $\xi_h : f \in C^\infty(P, \mathbf{R}) \longrightarrow \xi_h(f) = \{h, f\} \in C^\infty(P, \mathbf{R})$ verifies the following identities:

$$\begin{cases} \xi_h(af+bg) & = & a\xi_h(f)+b\xi_h(g) \\ \xi_h(f) & = & -\xi_f(h) \\ \xi_h(\{f,g\}) & = & \{\xi_h(f),g\}+\{f,\xi_h(g)\} \quad \text{(Jacobi)} \\ \xi_h(f\cdot g) & = & \xi_h(f)\cdot g+f\cdot\xi_h(g) \quad \text{(Leibniz).} \end{cases} \tag{8}$$

Let $SO(6)$ be the special orthogonal group, i.e. the group of all matrices X of type 6×6 with real coefficients such that $X^T\cdot X=I_6$ and $det(X)=1$. It is obvious that $SO(6)$ is a closed subgroup of the Lie group $\mathcal{M}_6(\mathbf{R})$, so it is a Lie group. Its Lie algebra is $so(6)$.

Let $so(6)^*$ be the dual space of $so(6)$. We have that $so(6)^*\cong so(6)$ and on other hand $so(6)$ is isomorphic with the Lie algebra $\mathbf{R}^{15}$ by $\widehat{f}:\mathbf{R}^{15}\longrightarrow so(6)$, see Proposition 1.

Then $so(6)^*$ has a canonical Lie-Poisson structure (see [6]), called the *minus Lie-Poisson structure* and it is determined by the matrix given by:

$$(\{u_i,u_j\}) \;=\; -(c_{i,j}^k u_k)=\begin{pmatrix} U_{11} & U_{12} \\ -U_{12}^T & U_{22}\end{pmatrix}, \tag{9}$$

with $U_{11}\in\mathcal{M}_9(\mathbf{R}), U_{12}\in\mathcal{M}_{9\times 6}(\mathbf{R}), U_{22}\in\mathcal{M}_6(\mathbf{R}),$ where

$$U_{11} \;=\; \begin{pmatrix} 0 & u_6 & u_7 & u_8 & u_9 & u_2 & -u_3 & -u_4 & -u_5 \\ -u_6 & 0 & u_{10} & u_{11} & u_{12} & u_1 & 0 & 0 & 0 \\ -u_7 & -u_{10} & 0 & u_{13} & u_{14} & 0 & u_1 & 0 & 0 \\ -u_8 & -u_{11} & -u_{13} & 0 & u_{15} & 0 & 0 & u_1 & 0 \\ -u_9 & -u_{12} & -u_{14} & -u_{15} & 0 & 0 & 0 & 0 & u_1 \\ -u_2 & -u_1 & 0 & 0 & 0 & 0 & u_{10} & u_{11} & u_{12} \\ u_3 & 0 & -u_1 & 0 & 0 & -u_{10} & 0 & u_{13} & u_{14} \\ u_4 & 0 & 0 & -u_1 & 0 & -u_{11} & -u_{13} & 0 & u_{15} \\ u_5 & 0 & 0 & 0 & -u_1 & -u_{12} & -u_{14} & -u_{15} & 0 \end{pmatrix},$$

$$U_{12} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -u_3 & -u_4 & -u_5 & 0 & 0 & 0 \\ u_2 & 0 & 0 & -u_4 & -u_5 & 0 \\ 0 & u_2 & 0 & u_3 & 0 & -u_5 \\ 0 & 0 & u_2 & 0 & u_3 & u_4 \\ -u_7 & -u_8 & -u_9 & 0 & 0 & 0 \\ u_6 & 0 & 0 & -u_8 & -u_9 & 0 \\ 0 & u_6 & 0 & u_7 & 0 & -u_9 \\ 0 & 0 & u_6 & 0 & u_7 & u_8 \end{pmatrix},$$

$$U_{22} \;=\; \begin{pmatrix} 0 & u_{13} & u_{14} & -u_{11} & -u_{12} & 0 \\ -u_{13} & 0 & u_{15} & u_{10} & 0 & -u_{12} \\ -u_{14} & -u_{15} & 0 & 0 & u_{10} & u_{11} \\ u_{11} & -u_{10} & 0 & 0 & u_{15} & -u_{14} \\ u_{12} & 0 & -u_{10} & -u_{15} & 0 & u_{13} \\ 0 & u_{12} & -u_{11} & u_{14} & -u_{13} & 0 \end{pmatrix}$$

and $c_{i,j}^k,\ i,j,k=\overline{1,15}$ are the structure constants of $so(6)$, given by Proposition 1.

It follows that the pair $(so(6)^*, \{\cdot,\cdot\})$ is a Poisson manifold.

4. THE SYMPLECTIC INTEGRATION OF $so(6)^*$

For basic notions, results and references about Lie groupoids and symplectic groupoids we refer the reader to [2], [9], [11].

Let $(G,\alpha,\beta,\mu,\varepsilon,\iota;G_0)$ be a *Lie groupoid over* G_0 or a *Lie groupoid with the base* G_0 (see [9]). We recall that $(G;G_0)$ is a pair of manifolds equipped with two surjective submersions $\alpha,\beta : G \to G_0$ (the *source* and the *target*), a differentiable map $\mu : G_{(2)} = \{(x,y)\in G\times G \mid \beta(x)=\alpha(y)\} \longrightarrow G,\ (x,y)\longrightarrow \mu(x,y)=xy$ (*partial multiplication law*), an injective differentiable map $\varepsilon : G_0 \to G,\ u \to \varepsilon(u)$ (*inclusion map*) and a differentiable map $\iota : G\longrightarrow G, x\longrightarrow \iota(x)=x^{-1}$ (*inversion map*).

These maps satisfy the following algebraic axioms generalizing those of group :

• $\alpha(xy)=\alpha(x)$ and $\beta(xy)=\beta(y)$, for all $(x,y)\in G_{(2)}$;

• (*associativity*) $(xy)z = x(yz)$, in the sense that , if one side of the equation is defined so is the other and then they are equal ;

• (*identities*) for each $x\in G$ we have $(\varepsilon(\alpha(x)),x)\in G_{(2)}, (x,\varepsilon(\beta(x)))\in G_{(2)}$ and $\varepsilon(\alpha(x))\cdot x = x\cdot\varepsilon(\beta(x)) = x$;

• (*inverses*) for each $x\in G$ we have $(x^{-1},x)\in G_{(2)},\ (x,x^{-1})\in G_{(2)},\ x\cdot x^{-1} = \varepsilon(\beta(x))$ and $x^{-1}\cdot x = \varepsilon(\alpha(x))$.

The subset $\varepsilon(G_0)$ of G is the *set of units* of the groupoid G over G_0.

Example 1. Let be the cotangent bundle $T^*SO(6)$ of the manifold $SO(6)$ and $\pi : T^*SO(6) \to SO(6)$ its projection. Then the addition in the fibers defines a Lie groupoid structure on $T^*SO(6)$ for which $G_0 = SO(6),\ \alpha=\beta=\pi, \iota : T^*SO(6) \to T^*SO(6)$ is the multiplication by -1 and $\varepsilon : SO(6) \to T^*SO(6)$ is the zero section. □

A *symplectic groupoid* is a Lie groupoid $(G,\alpha,\beta,\mu,\varepsilon,\iota;G_0)$ endowed with a symplectic structure ω on G for which the graph $\{\ (x,y,\mu(x,y))\in G\times G\times G \mid (\forall)\ (x,y)\in G_{(2)}\ \}$ of the groupoid multiplication μ is a Lagrangian submanifold of $(G\times G\times G, \omega\oplus\omega\oplus(-\omega))$.

A symplectic groupoid is denoted by (G,ω).

This interesting class of groupoids, which was introduced in [2], arises in the integration of arbitrary Poisson manifolds.

Example 2. On the Lie groupoid $T^*SO(6)$ there exists a canonical symplectic structure. For this, we can define the 1- form θ on $T^*SO(6)$ by $\theta(v)(g) = v(T\pi(g))$, where $v\in T^*SO(6),\ g\in T_v(T^*SO(6))$, and $T\pi : T(T^*SO(6)) \to TSO(6)$ is the tangent map to $\pi : T^*SO(6) \to SO(6)$. Then we define $\omega = d\theta$ so the pair $(T^*SO(6), \omega = d\theta)$ is a symplectic manifold.

It is easy to prove that this symplectic structure given on $T^*SO(6)$ is compatible with the groupoid structure of $T^*SO(6)$ over $SO(6)$, see Example 1.

It follows that $(T^*SO(6), \omega = d\theta)$ is a symplectic groupoid over $SO(6)$. □

A *symplectic integration* of a Poisson manifold (M,Λ) (see [12]), is a symplectic groupoid (G,ω) which realizes the Poisson manifold (M,Λ); i.e. such that the base G_0 with the induced Poisson structure Λ_0 is isomorphic to (M,Λ).

Theorem 1. *The pair* $(T^*SO(6),\omega=d\theta)$ *is a symplectic groupoid over* $so(6)^*$.

Proof. Let L_g and R_g be the left and right translations by g in $SO(6)$. These actions can be lifted to left and right actions on $T^*SO(6)$ as follows. Define
$L:(g,u_h)\in SO(6)\times T^*SO(6)\ \rightarrow\ L(g,u_h)=(T_{gh}L_{g^{-1}})^*u_h\in T^*SO(6)$
$R:(g,u_h)\in SO(6)\times T^*SO(6)\ \rightarrow\ R(g,u_h)=(T_{hg}R_{g^{-1}})^*u_h\in T^*SO(6)$.

These two commuting actions have the following Ad^*-equivariant momentum maps:
$J_L:u_h\in T^*SO(6)\ \rightarrow\ J_L(u_h)=(T_eR_h)^*u_h\in so(6)^*$ and
$J_R:u_h\in T^*SO(6)\ \rightarrow\ J_R(u_h)=(T_eL_h)^*u_h\in so(6)^*$.

Then $T^*SO(6)$ is a groupoid over $so(6)^*$ with J_R,J_L as the source and target maps.

Using the canonical identification $T^*SO(6)\cong SO(6)\times so(6)^*$ by right translations and the notations $\delta\nu\in T_\nu SO(6)$, $\delta_g h=T(R_h)\delta_g$ and $\nu\circ g=\nu\circ Ad(g)=Ad(g)^*\nu$, we may describe the symplectic groupoid structure as follows:
$\alpha(\nu,g)=\nu$; $\beta(\nu,g)=\nu\cdot g$; $\varepsilon(\nu)=(\nu,e)$; $\mu((\nu,g),(\nu\cdot g,h))=(\nu,gh)$ and
$\iota(\nu,g)=(\nu,g)^{-1}=(\nu\cdot g,g^{-1})$.

The symplectic structure $\omega=d\theta$ in this representation is given by :
$\omega((\delta\nu,\delta g),(\delta\nu',\delta g'))=<\delta\nu',\delta g\cdot g^{-1}>-<\delta\nu,\delta g'\cdot(g')^{-1}>=<\nu,[\delta g\cdot g^{-1},\delta g'\cdot(g')^{-1}]>$,
where $<\cdot,\cdot>$ is the pairing between $so(6)^*$ and $so(6)$. □

Theorem 2. *Let* $(T^*SO(6),\omega=d\theta)$ *the symplectic groupoid over* $so(6)^*$*. Then the induced Poisson structure on* $so(6)^*$ *is exactly the canonical Lie-Poisson structure on it.*

Proof. We have that the pair $(so(6)^*,\{\cdot,\cdot\})$ is a Poisson manifold (see Section 2).

The dual space $so(6)^*$ carries a canonical Lie-Poisson structure defined as follows. For $f,g\in C^\infty(so(6)^*,\mathbf{R})$ define

$$\{f,g\}_{LP}(v)=-<v,[\frac{\delta f}{\delta u},\frac{\delta g}{\delta u}]> \tag{10}$$

where $[\cdot,\cdot]$ is the standard Lie bracket on $so(6)$ and the element $\frac{\delta f}{\delta u}$ is given by:

$$Df(u)\cdot w=<w,\frac{\delta f}{\delta u}>,\ \text{ for all }\ u,w\in so(6)^*,$$

(here Df is the derivative of f).

In fact, the bracket $\{\cdot,\cdot\}_{LP}$ is the one induced on $so(6)^*$ by identifying $C^\infty(so(6)^*,\mathbf{R})$ with left invariant functions on $T^*SO(6)$. In terms of a basis $\{e_i\}$ and its dual basis $\{e^i\}$ with $u=u_ie^i$ the formulas (10) becomes

$$\{f,g\}_{LP}(u)=-\sum_{i,j,k}c^k_{i,j}\frac{\partial f}{\partial u_i}\frac{\partial g}{\partial u_j} \tag{11}$$

where $c^k_{i,j}$ are the structure constants of the Lie algebra $so(6)$.

Using the isomorphisms $T^*SO(6) \cong SO(6) \times so(6)^* \cong SO(6) \times \mathbf{R}^{15}$ (see Proposition 2), it is easy to deduce that the induced Poisson structure $\{\cdot,\cdot\}_0$ on $so(6)^*$ is in fact the canonical Lie-Poisson structure $\{\cdot,\cdot\}_{LP}$ on $so(6)^*$. $\square$

Theorem 3. *The symplectic integration of the Poisson manifold* $(so(6)^*, \{\cdot,\cdot\})$ *is the symplectic groupoid* $(T^*SO(6), \omega = d\theta)$ *over* $so(6)^*$.

Proof. By Theorem 1, the pair $(T^*SO(6), \omega = d\theta)$ is a symplectic groupoid over $so(6)^*$. By Theorem 2, the induced Poisson structure on $so(6)^*$ is isomorphic to $(so(6)^*, \{\cdot,\cdot\})$.

It follows that $(T^*SO(6), \omega = d\theta)$ is a symplectic integration of the Poisson manifold $(so(6)^*, \{\cdot,\cdot\})$. $\square$

5. Geometric prequantization of $so(6)^*$

Let us consider the following diagram:

$$\begin{pmatrix} so(6)^* \\ \{\cdot,\cdot\}_{LP} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}_0 \\ \delta^0 \end{pmatrix}$$

where in the right hand $\mathcal{H}_0$ is a Hilbert space and δ^0 is a map which assigns to each $f \in C^\infty(so(6)^*, \mathbf{R})$ a self-adjoint operator $\delta^0_f : \mathcal{H}_0 \to \mathcal{H}_0$ and in the left hand $so(6)^*$ is the dual of Lie algebra $so(6)$ togheter with its canonical Lie-Poisson structure.

The arrow left to right is called *prequantization*; i.e., it is a procedure to derive from the classical dates $(so(6)^*, \{\cdot,\cdot\}_{LP})$ and the quantum dates $(\mathcal{H}_0, \delta^0)$ such that the following conditions, called *Dirac conditions*, to be satisfied:

(D_1) $\delta^0_{f+g} = \delta^0_f + \delta^0_g$;

$(D_2$ $\delta^0_{af} = a \cdot \delta^0_f$;

(D_3) $\delta^0_{Id_{so(6)^*}} = Id_{\mathcal{H}_0}$;

(D_4) $[\delta^0_f, \delta^0_g] = i\hbar\delta^0_{\{f,g\}_{LP}}$

for each $f, g \in C^\infty(so(6)^*, \mathbf{R})$ and for each $a \in \mathbf{R}$, where $\hbar$ is the Planck's constant divided by 2π.

Let $L : SO(6) \times SO(6) \to SO(6)$ be the action of $SO(6)$ on itself by right translations and L^{T^*} its lift to $T^*SO(6)$. This action has the momentum map$J : T^*SO(6) \to so(6)^*$ given by:

$$(J(u_g))(\xi) = u_g(TR_g(\xi))$$

which is a Poisson map; see, Marsden and Raţiu ([10]).

It is well known that $(T^*SO(6), \omega = d\theta)$ is a quantizable manifold from the geometric point of view (since ω is a exact form); see Puta ([12]). Then its Hilbert representation is

$$\mathcal{H}^\omega = L^2(T^*SO(6), \mathbf{C})$$

where $L^2(T^*SO(6), \mathbf{C})$ denotes the Hilbert space of complex-valued functions defined on $T^*SO(6)$ which are square integrable and the prequantum operator

$\delta^\omega : \mathcal{H}^\omega \ \rightarrow \ \mathcal{H}^\omega$ is given by:

$$\delta^\omega_f = -i\hbar X_f - \theta(X_f) + f,$$

where X_f is the vector field which is canonically associated to f.

For $\mathcal{H}$ and δ^0_f we take

$$\mathcal{H}_0 = \mathcal{H}^\omega = L^2(T^*SO(6), \mathbf{C}) \ \text{ and } \ \delta^0_f = \delta^\omega_{f\circ\alpha} \ \text{ for each } \ f \in C^\infty(\mathbf{R}^{15}, \mathbf{R}). \qquad (12)$$

Theorem 4. *The pair $(\mathcal{H}_0, \delta^0)$ where $\mathcal{H}_0$ and δ^0 are given by (12), gives rise to a prequantization of the Poisson manifold $so(6)^* \cong \mathbf{R}^{15}$.*

Proof. For the proof we shall verify Dirac's conditions $(D_1) - (D_4)$. The conditions $(D_1) - (D_3)$ are easily verified. For the condition (D_4) we have succesively:

$$[\delta^0_f, \delta^0_g] = [\delta^\omega_{f\circ\alpha}, \delta^\omega_{g\circ\alpha}] = i\hbar\delta^\omega_{\{f\circ\alpha, g\circ\alpha\}_\omega} =$$

$$= i\hbar\delta^\omega_{\{f,g\}_{LP}\circ\alpha} = i\hbar\delta^0_{\{f,g\}_{LP}}, \ (\forall) \ f, g \in C^\infty(\mathbf{R}^{15}, \mathbf{R}),$$

where we have used the property of α to be a Poisson map, that is
$\{f\circ\alpha, g\circ\alpha\}_\omega = \{f, g\}_{LP} \circ \alpha.$ □

Using the same arguments as in the paper of Chernoff ([1]) with obvious modifications it is easy to prove the following theorem.

Theorem 5. *Let $\mathcal{O}(L^2(T^*SO(6), \mathbf{C}))$ be the space of self-adjoint operators defined on $L^2(T^*SO(6), \mathbf{C})$. Then the map:*

$$f \in C^\infty(so(6)^*, \mathbf{R}) \ \rightarrow \ \delta^0_f \in \mathcal{O}(L^2(T^*SO(6), \mathbf{C}))$$

gives rise to an irreductible representation of $C^\infty(so(6)^*, \mathbf{R})$ onto $\mathcal{O}(L^2(T^*SO(6), \mathbf{C}))$.

□

For more details concerning the geometric quantization and its applications in geometry and quantum mechanics, see [8], [12], [13], [15].

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