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The polynomial function of order 3 revisited

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ABSTRACT. We study the symmetry point of the parabola of order three, we characterize the roots of the polynomial of order three, and we give some conditions for the position of the real roots related to given real numbers.

It is well known that the second order polynomial function

$$f:\mathbb{R}\to\mathbb{R}, f(x)=ax^2+bx+c$$

shortly the second order function, has got its associated graph a parabola (of order 2).

The symmetry properties of this function are known as well by most of the secondary school students as well.This function has got as symmetry axis the line $x=-\dfrac{b}{2a}$, in other words

$$f(-\frac{b}{2a}-x)=f(-\frac{b}{2a}+x)$$

is true for all real x.

This symmetry is used (see for example [1], or [2]) to establish the relation between the coefficients and the position of the roots of the equation of order 2

$$ax^2+bx+c=0$$

related to one or two fixed numbers.

The naturally arising question is if the polynomial function of order 3 has got any symmetry or not? We will see in the sequel that the function.

$$f:\mathbb{R}\to\mathbb{R}, f(x)=ax^3+bx^2+cx+d$$

has a symmetry point, let us denote it by $S(x_s,y_s)$, for which the relation

$$y_s=f(x_s)=\frac{f(x_s-x)+f(x_s+x)}{2}$$

holds for all real x.

Lemma 1. *The function* $f:R\to\mathbb{R}$, $f=ax^3+bx^2+cx+d$ *has got the symmetry point* $S=(\dfrac{b}{3a},f(-\dfrac{b}{3a}))$.

Proof. It is sufficient to prove for all real x:

$$f(-\frac{b}{3a})-f(-\frac{b}{3a}-x)=f(-\frac{b}{3a}+x)-f(-\frac{b}{3a})$$

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or

$$f(-\frac{b}{3a}) = \frac{f(-\frac{b}{3a}+x)+f(-\frac{b}{3a}-x)}{2}$$

We have to compute the following three values:

$$f(-\frac{b}{3a}) = a(-\frac{b}{3a})^3 + b(-\frac{b}{3a})^2 + c(-\frac{b}{3a}) + d$$

$$f(-\frac{b}{3a}+x) = a(-\frac{b}{3a}+x)^3 + b(-\frac{b}{3a}+x)^2 + c(-\frac{b}{3a}+x) + d$$

$$f(-\frac{b}{3a}-x) = a(-\frac{b}{3a}-x)^3 + b(-\frac{b}{3a}-x)^2 + c(-\frac{b}{3a}-x) + d.$$

Computed details show that:

$$f(-\frac{b}{3a}) = -a\frac{b^3}{27a^3} + b\frac{b^2}{9a^2} - c\frac{b}{3a} + d$$

$$\begin{aligned} f(-\frac{b}{3a}+x) &= -a\frac{b^3}{27a^3} + 3a\frac{b^2}{9a^2}x - 3a\frac{b}{3a}x^2 + ax^3 + \\ &\quad +b\frac{b^2}{9a^2} - 2b\frac{b}{3a}x + bx^2 - c\frac{b}{3a} + cx + d \end{aligned}$$

and

$$\begin{aligned} f(-\frac{b}{3a}-x) &= -a\frac{b^3}{27a^3} - 3a\frac{b^2}{9a^2}x - 3a\frac{b}{3a}x^2 - ax^3 + \\ &\quad +b\frac{b^2}{9a^2} + 2b\frac{b}{3a}x + bx^2 - c\frac{b}{3a} - cx + d. \end{aligned}$$

Finally if summing up the last two lines we will have:

$$f(-\frac{b}{3a}+x) + f(-\frac{b}{3a}-x) = -2a\frac{b^3}{27a^3} + 2\frac{b^3}{9a^2} - 2\frac{bc}{3a} + 2d = 2f(-\frac{b}{3a})$$

and this ends the proof. □

The discussion of the position of the roots of the third order equation

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

related to one or more numbers is not as simple as in the second order case. The above stated symmetry makes possible to solve most of these problems.

Let us take the real function

$$f(x) = ax^3 + bx^2 + cx + d$$

and the first order derivative of it:

$$f'(x) = 3ax^2 + 2bx + c.$$

Denote by α and β the roots of the above first derivative.

Lemma 2. *In the case*

$$D = 4b^2 - 12ac > 0,$$

α and β are different real numbers, and we can distinguish 3 cases:

1. The roots of the equation (1) are all real and different if and only if :

$$f(\alpha)f(\beta) < 0$$

2. The roots are real but not different if and only if:

$$f(\alpha)f(\beta) = 0$$

3. The equation (1) has got only one real root (and two conjugated complex roots) if and only if:

$$f(\alpha)f(\beta) > 0$$

Remark 3. The expression $H = f(\alpha)f(\beta)$ is symmetric in α and β

Indeed,

$$H = (a\alpha^3 + b\alpha^2 + c\alpha + d)(a\beta^3 + b\beta^2 + c\beta + d)$$

$$H = a^2\alpha^3\beta^3 + ab\alpha^2\beta^2(\alpha+\beta) + ac\alpha\beta(\alpha^2+\beta^2) + ad(\alpha^3+\beta^3) + b^2(\alpha^2+\beta^2) +$$

$$+bc\alpha\beta(\alpha+\beta) + bd(\alpha^2+\beta^2) + c^2\alpha\beta + cd(\alpha+\beta) + d^2,$$

consequently it can be expressed with the sum and the product of α and β.

We have

$$\alpha + \beta = -\frac{2b}{3a}, \text{ and } \alpha\beta = \frac{c}{3a}.$$

The computations lead to

$$H = \frac{4ac^3 - b^2c^2 + 4b^3d - 18abcd + 27a^2d^2}{27a^2}$$

which can be written

$$H = \frac{3(bc - 3ad)^2 + 4(ac^3 - b^2c^2 + b^3d)}{27a^2}$$

Conclusion 4. As a consequence we can state that the polynomial

$$f(x) = ax^3 + bx^2 + cx + d$$

has got:

(a) three different real roots for $H < 0$

(b) real, but not different roots for $H = 0$

(c) One real, and two conjugated complex roots for $H > 0$.

The Cardano formula to solve the polynomial equation of order 3

$$x^3 + px + q = 0 \tag{2}$$

states that the roots are of the form:

$$\begin{aligned} x_1 &= u + v \\ x_2 &= \varepsilon u + \varepsilon^2 v \\ x_3 &= \varepsilon^2 u + \varepsilon v \end{aligned}$$

where

$$\varepsilon = -\frac{1}{2} - \frac{q}{2} + i\frac{\sqrt{3}}{2}$$

and

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}},$$

$$v = \sqrt[3]{-\frac{q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}.$$

Using the proper notations the above expression H becomes:

$$H = \frac{4p^3 + 27q^2}{27}$$

or

$$H = 4\left[(\frac{p}{3})^3 + (\frac{q}{2})^2\right]$$

Conclusion 5. In [3] it is shown the way that the roots of the equation (2) are characterized by the same conditions as before:
(a) three different real roots for $H < 0$
(b) real, but not different roots for $H = 0$
(c) One real, and two conjugated complex roots for $H > 0$.

The proof in [3] is in a pure algebric way, but this method is more complicated due to the complex numbers involved there, and the real roots appear written as depending on complex numbers.

Application.
The discussion of the position of the real roots of the third order equation (1) becomes possible in most of the cases.
Let us write again the given equation

$$ax^3 + bx^2 + cx + d = 0 \qquad (1)$$

and take the real numbers m, n, and use the notions :

$$f(x) = ax^3 + bx^2 + cx + d$$

and

$$H = \frac{3(bc - 3ad)^2 + 4(ac^3 - b^2c^2 + b^3d)}{27a^2}$$

and the binary logical operations $\vee$(OR), $\wedge$(AND).

Case 1. $(x_1 > m) \wedge (x_2 > m) \wedge (x_3 > m)$ will be true only and only if:

$$(H \leq 0) \wedge (-\frac{b}{3a} > m) \wedge [(af(m) < 0) \vee (af'(m) > 0)]$$

Case 2. $(x_1 < m) \wedge (x_2 < m) \wedge (x_3 < m)$ will be true only and only if:

$$(H \leq 0) \wedge (-\frac{b}{3a} < m) \wedge [(af(m) > 0) \vee (af'(m) < 0)] \,.$$

Case 3. $(x_1 < m) \wedge (x_2 > m) \wedge (x_3 > m)$ will be true only and only if:

$$(H \leq 0) \wedge (af(m) > 0) \wedge \left[(m < -\frac{b}{3a}) \vee (af'(m) < 0)\right].$$

Case 4. $(x_1 < m) \wedge (m < x_2 < n) \wedge (n < x_3)$ will be true only and only if:

$$(af(m) > 0) \wedge (af(n) < 0).$$

Case 5. $(x_1, x_2, x_3 \in [m, n])$ will be true only and only if:

$$(H \leq 0) \wedge (af(m) < 0) \wedge (af(n) > 0) \wedge (m < -\frac{b}{3a} < n) \wedge (f'(m) \cdot f'(n) > 0).$$

Case 6. $(x_2 - x_1 > x_3 - x_2)$ will be true only and only if:

$$(a \cdot f(-\frac{b}{3a}) > 0) \wedge (H < 0).$$

Case 7. $(x_2 - x_1 < x_3 - x_2)$ will be true only and only if:

$$(a \cdot f(-\frac{b}{3a}) < 0) \wedge (H < 0).$$

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