CREATIVE MATH. & INF.
15 (2006), 41 - 46

New convergence rate of Fibonacci approach to ordinary differential equations

J. O. OMOLEHIN

ABSTRACT. The convergence rate of Fibonacci numerical approach to the solution of ordinary differential equation is guaranteed under linear boundary valued problem.

1. INTRODUCTION

The Tau method for the numerical solution of O.D.E's requires that the coefficients and the right hand side of the differential equation be polynomials. However, method of selected points removes this restriction. See Lanczos [2]. The choice of Chebyshevs perturbation term is lost in the method of selected points. Okunuga and Onumaniyi [1] retained the choice of Chebyshev perturbation term and they extended their work to ordinary differential equation with non-polynomial coefficients. The present work considers a new collocation approach to Tau method. The approximating perturbation term is what is called Fibonacci polynomial. Another interesting point in the new method is the choice of one perturbation term against multiple choices of perturbation terms in collocation Tau method [1]. The new method also allows the result equations to be the same number as the number of the unknowns so that the system of equations involved can be solved uniquely.

2. COLLOCATION TAU METHOD

Recall the tau method for general problem. That is we consider the linear m^{th} order ordinary differential equation.

$$f(x) = q_m(x)\frac{d^m y(x)}{dx^m} + ... + q_0(x)y(x) \tag{1}$$
$$a \leq x \leq b$$

where $q_j(x), j = 1, 2, ..., m$ are not necessarily polynomials. The solution of (1) is approximated by a power series of the form

$$y(x) = \sum_{i=0}^{n} a_i x^i$$

On expanding the last equation we have

$$y_n(x) = a_0 + a_1 x + ... + a_n x^n \tag{2}$$

Received: 13.06.2006. In revised form: 1.10.2006.
2000 *Mathematics Subject Classification.* 34K28.
Key words and phrases. Fibonacci, collocation, numerical, tau method.

substitute (2) in (1) to obtain

$$q_m(x)\frac{d^m y_n(x)}{dx^m} + ... + q_0(x)y_n(x) = f(x) + R_n(x) \tag{3}$$

where $R_n(x)$ is the residual. For polynomials coefficient $q_j(x), j = 0, 1, ..., m$ the residual $R_n(x)$ is a polynomial which is replaced by a finite sum of chebyshev polynomials involving some parameters τ. If the coefficients $q_j(x), j = 0, 1, ..., n$ are non polynomial, Lanczos [2] considers

$$R_n(x) = \left(\tau_1 + \tau_2 x + ... + \tau_m x^{m-1}\right)(x)T_{n-m+1}(x) \tag{4}$$

where

$$T_n(x) = cos\left(ncos^{-1}(x)\right), -1 \leq x \leq 1$$

$$f(x) = q_n(Z_i)\frac{d^m y_n(Z_i)}{dx^m} + ... + q_0(Z_i)y_n(Z_i) \tag{5}$$

The parameters $\tau_i, i = 0, 1, ..., m$ in (5) are naturally eliminated from (4) by the choice of Z_i. The collocation tau method (1) considers and approximate value for the residual R_n in the form:

$$R_n = \tau_1 T_n^*(x) + \tau_2 T_{n-1}^*(x) + ... + \tau_m T_{n-m+1}^*(x) \tag{6}$$

Substitute (6) in (3) and collocate the result equation at

$$x_i = ih,\ i = 1, 2, ..., n+1,\ h = \frac{(b-a)}{(n+2)} \tag{7}$$

Therefore

$$\begin{aligned} q_m(x_i)\frac{d^m y_n(x)}{dx_m} + q_0(x_i)y_n(x) = \\ = f(x_i) + \tau_1 T_n^*(x_i) + \tau_2 T_{n-1}^*(x_i) + ... + \tau_m T_{n-m+1}^*(x_i) \end{aligned} \tag{8}$$

3. Main Result

This new approach makes use of a sequence of positive integers known as the Fibonacci numbers. These numbers are defined by the relations:

$$\begin{aligned} F_0 = F_1 = 1 \\ F_n = F_{n-1} + F_{n-2},\ n \geq 2 \end{aligned} \tag{9}$$

and therefore the sequence begins with $1, 1, 2, 3, 5, 13, 21, 34, ...$
We now define our residual as:

$$r_n(x) = \tau F_p(x) \tag{10}$$

where $F_p(x)$ is the Fibonacci polynomial defined by

$$F_p(x) = F_0 + F_1 x + F_2 x^2 + ... \tag{11}$$

where $F_i, i = 0, ..., n$ are as previously defined in equation (9) above. We can now substitute equation (11) into (3) and collocate the result equation at

$$x_i = ih, i = 1, 2, ..., n+1, h = \frac{(b-a)}{(n+2)}$$

to obtain

$$q_m(x_i)\frac{d^m y_n}{dx^m} + q_0(x_i)y_n(x_i) = F(x_i) + \tau F_p(x_i) \tag{12}$$

Equation (12) coupled with the boundary conditions will result in $n+2$ unknowns. The values of τ and $a_i, i = 0, 1, 2, ..., n$ can now be determined uniquely.

4. Numerical computations

Two problems are now considered.

Problem 1.

Consider the problem

$$y^{''}(x) - y(x) = 0,\ 0 \le x \le 1,\ y(0) = 1, y(1) = 3 \tag{13}$$

Solution:

In order to solve this problem with the new approach, an approximate power series solution is assumed in the form:

$$y_n(x) = \sum_{i=0}^{n} a_i x^i \tag{14}$$

Equation (15) is now substituted to the slightly perturbed equation (14)

$$y_n^{''}(x) - y_n(x) = \tau F_p \tag{15}$$

Now

$$\begin{aligned} y_n(x_0) &= \sum_{i=0}^{n} a_i x^i \\ y_n^{'} &= \sum_{i=0}^{n} i a_i x^{i-1} \\ y_n^{''} &= \sum_{i=0}^{n} (i)(i-1) a_i x^{i-2} \\ F_p &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 \end{aligned} \tag{16}$$

Let us take $n = 5$, substitute equation (16) into equation (15), we obtain

$$\begin{aligned} (2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5) = \\ = \tau(1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5) \end{aligned} \tag{17}$$

Substitute $y_5(0) = 0$ in equation (17), we obtain

$$a_0 + a_1 \times 0 + a_2 \times 0 + a_3 \times 0 + a_4 \times 0 + a_5 \times 0 \ = \ 1$$

It implies that $a_0 = 1$.

At $y_5(1) = 3$, equation (17) becomes

$$a_0 + a_1 \times 1 + a_2 \times 1 + a_3 \times 1 + a_4 \times 1 + a_5 \times 1 = 3$$

i.e.

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 3$$

But $a_0 = 1$, this implies that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 2$$

Collocate equation is $x_i = \frac{(b-a)i}{(n+1)},\ i = 1, 2, ..., 5.$
Thus the collocation points are:

$$x \ = \ 1/6,\ 2/6,\ 3/6,\ 4/6,\ 5/6 \tag{18}$$

Collocating equation (17) at points in (18), coupling with the boundary conditions, we obtain seven equations with seven unknowns:

$$\begin{gathered}
a_0 = 1\\
a_1 + a_2 + a_3 + a_4 + a_5 = 2\\
-7776a_0 - 1296a_1 + 1536a_2 + 7740a_3 + 2586a_4 + 719a_5 = \tau(9650)\\
-7776a_0 - 3888a_1 + 13608a_2 + 22356a_3 + 22842a_4 + 19197a_5 = \tau(22842)\\
-7776a_0 - 2592a_1 + 14688a_2 + 15264a_3 + 10272a_4 + 572a_5 = \tau(13696)\\
-7776a_0 - 5184a_1 + 12096a_2 + 28800a_3 + 39936a_4 + 45056a_5 = \tau(42656)\\
-7776a_0 - 6480a_1 + 10152a_2 + 34380a_3 + 61050a_4 + 86875a_5 = \tau(82306)
\end{gathered}$$

Eliminating a_0 the system of the equation now becomes:

$$\begin{gathered}
a_1 + a_2 + a_3 + a_4 + a_5 = 2\\
-1296a_1 + 1536a_2 + 7740a_3 + 2586a_4 + 719a_5 - \tau(9650) = 7776\\
-3888a_1 + 13608a_2 + 22356a_3 + 22842a_4 + 19197a_5 - \tau(22842) = 7776\\
-2592a_1 + 14688a_2 + 15264a_3 + 10272a_4 + 572a_5 - \tau(13696) = 7776\\
-5184a_1 + 12096a_2 + 28800a_3 + 39936a_4 + 45056a_5 - \tau(42656) = 7776\\
-6480a_1 + 10152a_2 + 34380a_3 + 61050a_4 + 86875a_5 - \tau(82306) = 7776
\end{gathered}$$

which is in form of:

$$AX = B$$

where

$$X = (a_1, a_2, a_3, a_4, a_5)^T \text{ and } B = (2, 7776, 7776, 7776, 7776, 7776)^T$$

A Gaussian elimination technique is used to solve the result system of equations. The constants $a_i(i \ = \ 0, 1, 2, ..., n)$ and τ are now substituted back to obtain the required solution. Our error is defined by:

$$\mid y_{n-1}(x) - y_n(x) \mid$$

The numerical results are displayed in Table 1.

Problem 2.

$$2(1 + x)y^{''}(x) + y(x) = 0,\ y(0) = 1$$

The same processes are followed as in Problem 1 and the numerical results are displayed in Table 2.

Table 1:Numerical error for Problem 1

	$n = 5$	$n = 6$
X	ERROR	ERROR
0.1	.0018734	2.9841950E-004
0.2	0.0034658	0.005346
0.3	0.00047648	0.0007170
0.4	0.0057564	0.0008397
0.5	0.0064080	0.0008988
0.6	.0066518	0.00908874
0.7	0.0063673	0.0007973
0.8	0.0053651	0.0006205
0.9	0.003693	3.533363E-004
1	2.3841860E-007	0.000000

Table 2: Numerical error for Problem 2

	$n = 5$	$n = 6$
X	ERROR	ERROR
0.1	0.0169250	2.3841860E-007
0.2	0.0329049	9.0003010E-006
0.3	0.0479466	6.84855740E-005
0.4	0.0624208	2.8860570E-004
0.5	0.0767598	8.807E-004
0.6	0.0911549	2.1916E-003
0.7	0.1052536	4.7369E-003
0.8	0.1178574	9.2354E-002
0.9	0.1266185	1.66425E-002
1	0.1277382	2.81843E-002

5. CONCLUSION

It can be seen from Table 1 and Table 2 that the new collocation approach is favorably comparable with any other collocation method [2]. Take for instance [2], Lanczos used multiple τ in his solutions but only one τ is required for our perturbation term in the new method. The choice of Fibonacci polynomials is very easy as compared with the computations of shifted Chebyshev polynomial [1]. The convergence rate of our method is also assured based on the asymptotic convergence rate of Fibonacci search given by Wismer. For details see [9]. In the two examples considered it can be seen in Table 1 and Table 2 that the results are better at $n = 6$ than at $n = 5$. Our method gives the exact value for $x = 1.0$ when $n = 6$ for Problem 1. All these are very interesting.

Since this is the first attempt to introduce Fibonacci polynomials into the numerical solution of O.D.E.'s, to best of knowledge of the author, more work will continue and findings will be reported in due course.

References

[1] Okunuga C. A. A., Onumanyi P., *A collocation tau method for the solution of ordinary differential equations*, Journal of Mathematics Association of Nigeria, Vol. 17. No 1 (1986), 51-60

[2] Lanczos C., *Applied analysis, Prentice Hall*, New Jersey, 1956

[3] Fox L., *Chebyshev Methods for ordinary differential equations*, Comp. J. 4 (1962), 318-331

[4] Chen P.Y.P., Lanczos C., *Chebyshev reduction method for the solutions of boundary value problem*, Int. Num. Meth. Eng. 17, pp. 291-299, 1981

[5] Taiwo A. O., Onumanyi P., *P A collocation approximation of singularly perturbed second order differential equation*, Int. J. Computer Math. Vol. 39 (1990), 205-211

[6] Taiwo A. O., *A comparison between spline and collocation tau methods*, Int. J. Computer Math. Vol. 32 (2005), 1889-1899

[7] Ibiejugba M. A., Oumanyi P., *On control operators and some of its Applications*, Journal of Mathematical Analysis and Applications, Vol. 103 (1984), 31-47

[8] Omolehin J. O., Sinkaye L., *Solving Linear Quadratic Optimal Control Problems Using Legendre Polynomials Approximation Method*, A.M.S.E., 42(5) (2005), 23-34

[9] Wismer D. A., Charttergy R., *Introduction to nonlinear optimization. A problem solving approach*, North - Holland - New York, 1978

MATHEMATICS DEPARTMENT
UNIVERSITY ILORIN
ILORIN, PMB 1515
KWARA STATE, NIGERIA
E-mail address: omolehin_joseph@yahoo.com