CREATIVE MATH. & INF.
15 (2006), 47 - 52

About $T_{m,s}(x)$ polynomials

OVIDIU T. POP

ABSTRACT. In this paper we give a new proof of some relations verified by $T_{m,s}(x)$ polynomials.

1. INTRODUCTION

Remind some known notions and results (see [1], [3] or [6]).

Definition 1.1. For $m \in \mathbb{N}$, define

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \quad \forall\, k \in \{0,1,\dots,m\}, \quad \forall\, x \in [0,1]. \tag{1.1}$$

Definition 1.2. For $m \in \mathbb{N}$, define

$$T_{m,s}(x) = \sum_{k=0}^{m} (k-mx)^s p_{m,k}(x), \quad \forall\, s \in \mathbb{N}, \quad \forall\, x \in [0,1]. \tag{1.2}$$

Example 1.1. From Definition 1.2 we immediately have

$$T_{m,0}(x) = \sum_{k=0}^{m} p_{m,k}(x) = 1,\ T_{m,1}(x) = \sum_{k=0}^{m} (k-mx) p_{m,k}(x) = 0 \text{ and}$$

$$T_{m,2}(x) = \sum_{k=0}^{m} (k-mx)^2 p_{m,k}(x) = mx(1-x), \quad \forall\, x \in [0,1].$$

Lemma 1.1. *Let $m \in \mathbb{N}$ and $s \in \mathbb{N}^*$. Then*

$$T_{m,s+1}(x) = x(1-x)\left[T'_{m,s}(x) + msT_{m,s-1}(x)\right], \forall\, x \in [0,1]. \tag{1.3}$$

Example 1.2. Taking Lemma 1.1 and Example 1.1 into account, we obtain

$$\begin{aligned} T_{m,3}(x) &= mx(1-x)(1-2x) \quad \text{and} \\ T_{m,4}(x) &= 3m^2x^2(1-x)^2 + m\left[x(1-x) - 6x^2(1-x)^2\right], \quad \forall\, x \in [0,1]. \end{aligned}$$

The study of $T_{m,s}(x)$ polynomials is important because these polynomials appear in Voronovskaja's generalized theorem.The text of this theorem is:

Theorem 1.1. *Let $f : [0,1] \to \mathbb{R}$ be s times derivable function in the point $x \in [0,1]$, $s \in \mathbb{N}$, s even. Then*

$$\lim_{m\to\infty} m^{\frac{s}{2}} \left[\big(B_m f\big)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!} T_{m,i}(x) f^{(i)}(x) \right] = 0\,. \tag{1.4}$$

For Theorem 1.1 and her consequences see [2], [3] or [5].

Received: 10.02.2006. In revised form: 25.08.2006.
2000 *Mathematics Subject Classification.* 41A10, 41A36.
Key words and phrases. The $T_{m,s}(x)$ polynomials and generalized Voronvskaja's theorem.

2. MAIN RESULTS

Considering Example 1.1, Example 1.2 and Lemma 1.1, for $m, s \in \mathbb{N}$, $s \neq 1$ and $x \in [0,1]$, we have

$$T_{m,s}(x) = a^{(0)}_{m,s}(x)m^n + a^{(1)}_{m,s}(x)m^{n-1} + \cdots + a^{(n)}_{m,s}(x) \tag{2.1}$$

where $a^{(i)}_{m,s}(x)$ with $i \in \{0,1,\ldots,n\}$ are polynomials of variable x and

$$T_{m,s}(x) = b^{(0)}_{m,s}(m)x^p + b^{(1)}_{m,s}(m)x^{p-1} + \cdots + b^{(p)}_{m,s}(m)\,, \tag{2.2}$$

where again $b^{(j)}_{m,s}(m)$, with $j \in \{0,1,\ldots,p\}$ are polynomials of variable m.

Definition 2.1. a) If $\alpha \in [0,1]$ exists so that $a^{(0)}_{m,s}(\alpha) \neq 0$, then we say that the degree of m in $T_{m,s}(x)$ is n and we write $gr_m T_{m,s}(x) = n$;

b) If $\beta \in \mathbb{N}$ exists so that $b^{(0)}_{m,s}(\beta) \neq 0$, we say that the degree of x in $T_{m,s}(x)$ is p and write $gr_x T_{m,s}(x) = p$.

Example 2.1. Heeding the Examples 1.1 and 1.2 we have
$gr_m T_{m,2}(x) = 1, \quad gr_m T_{m,3}(x) = 1, \quad gr_m T_{m,4}(x) = 2,$
$gr_x T_{m,2}(x) = 2, \quad gr_x T_{m,3}(x) = 3$ and $gr_x T_{m,4}(x) = 4\,.$

Theorem 2.1. *Let $m \in \mathbb{N}^*$ and $s \in \mathbb{N}$,$s \neq 1$. Then*

$$gr_m T_{m,s}(x) = \left[\frac{s}{2}\right] \tag{2.3}$$

and

$$gr_x T_{m,s}(x) = s\,. \tag{2.4}$$

Proof. For $s = 0$, relations (2.3) and (2.4) are true. Let $s \in \mathbb{N}$, $s \geq 2$.
We will prove by induction after s that

$$\begin{aligned} T_{m,s}(x) &= a_{m,s}(x)m^{\left[\frac{s}{2}\right]} + p_{m,s}(m,x)\,, \\ gr_m T_{m,s}(x) &= \left[\frac{s}{2}\right], gr_m p_{m,s}(m,x) < \left[\frac{s}{2}\right], \quad \forall\, s \in \mathbb{N},\ s \geq 2\,, \end{aligned} \tag{2.5}$$

where $gr_m p_{m,s}(m,x)$ is considered according to Definition 2.1.

For $s = 2$ we have $T_{m,s}(x) = a_{m,2}(x)m + p_{m,2}(m,x)$, where $a_{m,2}(x) = x(1-x)$ and $p_{m,2}(m,x) = 0$. Assume that $T_{m,n}(x) = a_{m,n}(x)m^{\left[\frac{n}{2}\right]} + p_{m,n}(m,x)$, $gr_m T_{m,n}(x) = \left[\frac{n}{2}\right]$, $gr_m p_{m,n}(m,x) < \left[\frac{n}{2}\right]$, $\forall\, n \in \{2,3,\ldots,s\}$. From (2.5) and Lemma 1.1 it follows that

$$\begin{aligned} T_{m,s+1}(x) &= x(1-x)\left[T'_{m,s}(x) + msT_{m,s-1}(x)\right] = \\ &= x(1-x)\left\{a'_{m,s}(x)m^{\left[\frac{s}{2}\right]} + p'_{m,s}(m,x) + ms\big[a_{m,s-1}(x)m^{\left[\frac{s-1}{2}\right]} + \right.\\ &\left. + p_{m,s-1}(m,x)\big]\right\} = x(1-x)\left[a'_{m,s}(x)m^{\left[\frac{s}{2}\right]} + sa_{m,s-1}(x)m^{\left[\frac{s+1}{2}\right]} + \right.\\ &\left. + p'_{m,s}(m,x) + msp_{m,s-1}(m,x)\right] \end{aligned}$$

and hence

$$T_{m,s+1}(x) = a_{m,s+1}(x) m^{\left[\frac{s+1}{2}\right]} + p_{m,s+1}(m,x), \tag{2.6}$$

where

$$a_{m,s+1}(x) = \begin{cases} x(1-x)\left[a'_{m,s}(x) + s a_{m,s-1}(x)\right], & \text{if } s \text{ is even} \\ x(1-x) s a_{m,s-1}(x), & \text{if } s \text{ is odd} \end{cases} \tag{2.7}$$

and

$$p_{m,s+1}(m,x) = \begin{cases} x(1-x)\left[p'_{m,s}(m,x) + ms\, p_{m,s-1}(x)\right], & \text{if } s \text{ is even} \\ x(1-x)\left[a'_{m,s}(x) m^{\left[\frac{s}{2}\right]} + p'_{m,s}(m,x) + ms\, p_{m,s-1}(x)\right], \\ \text{if } s \text{ is odd}. \end{cases} \tag{2.8}$$

From (2.7) we have that $a_{m,s+1}(x)$ is not the identical null polynomial, thus there exists $\alpha \in [0,1]$ so that $a_{m,s+1}(\alpha) \neq 0$.

From (2.8) results that $gr_m p_{m,s+1}(m,x) < \left[\dfrac{s+1}{2}\right]$ so (2.3) holds.

We prove through induction after s that

$$T_{m,s}(x) = b_{m,s}(m) x^s + r_{m,s}(m,x), \tag{2.9}$$

$gr_x T_{m,s}(x) = s,\ gr_x r_{m,s}(m,x) < s$, $\forall\ s \in\mathbb{N}$, $s \geq 2$, where $gr_x r_{m,s}(m,x)$ is considered according to Definition 2.1.

If $s = 2$ we have that $T_{m,2}(x) = b_{m,2}(m)x^2 + r_{m,2}(m,x)$, where $b_{m,2}(m) = -m$ and $r_{m,2}(m,x) = mx$. Assume that

$$T_{m,n}(x) = b_{m,n}(m)x^n + r_{m,n}(m,x),\ gr_x T_{m,n}(x) = n,\ gr_x r_{m,n}(m,x) < n \tag{2.10}$$

for $\forall\ n \in \{2, 3, \ldots, s\}$.

Taking (2.10) and Lemma 1.1 into account we have

$$\begin{aligned} T_{m,s+1}(x) &= x(1-x)\left[T'_{m,s}(x) + ms\, T_{m,s-1}(x)\right] = \\ &= x(1-x)\Big\{ b_{m,s}(m) s x^{s-1} + r'_{m,s}(m,x) + \\ &+ ms\left[b_{m,s-1}(m)x^{s-1} + r_{m,s-1}(m,x)\right] \Big\}, \end{aligned}$$

so

$$T_{m,s+1}(x) = b_{m,s+1}(m) x^{s+1} + r_{m,s+1}(m,x), \tag{2.11}$$

where

$$b_{m,s+1}(m) = -s b_{m,s}(m) - ms b_{m,s-1}(m)$$

and

$$\begin{aligned} r_{m,s+1}(m,x) &= [s\, b_{m,s}(m) + ms\, b_{m,s-1}(m)]\, x^s + \\ &+ \left[r'_{m,s}(m,x) + ms\, r_{m,s-1}(m,x)\right] x(1-x). \end{aligned}$$

The relation (2.10) leads to the fact that $b_{m,s+1}(m)$ is not the identical null polynomial, so $\beta \in\mathbb{N}$ exists such that $b_{m,s+1}(\beta) \neq 0$. It also results from (2.11) that $gr_x r_{m,s+1}(m,x) < s+1$, hence (2.4) is true. □

Theorem 2.2. *Let $m \in \mathbb{N}^*$ and $s \in \mathbb{N}$. Then the following is true*

$$T_{m,s}(x) = [x(1-x)]^{\left[\frac{s}{2}\right]}\big(\alpha_{m,s}x + \beta_{m,s}\big)m^{\left[\frac{s}{2}\right]} + p_{m,s}(m,x)\,, \tag{2.10}$$

$$\alpha_{m,s} = \begin{cases} 0\,, & \textit{if } s \textit{ is even or } s = 1 \\ -(s-1)!! \sum\limits_{k=1}^{\left[\frac{s}{2}\right]} \dfrac{(2k-1)!!}{(2k-2)!!}, & \textit{if } s \textit{ is odd, } s \geq 3, \end{cases} \tag{2.11}$$

$$\beta_{m,s} = \begin{cases} 1, & \textit{if } s = 0 \\ 0, & \textit{if } s = 1 \\ (s-1)!!\,, & \textit{if } s \textit{ is even, } s \geq 2 \\ \dfrac{1}{2}(s-1)!! \sum\limits_{k=1}^{\left[\frac{s}{2}\right]} \dfrac{(2k-1)!!}{(2k-2)!!}, & \textit{if } s \textit{ is odd, } s \geq 3 \end{cases} \tag{2.12}$$

and $gr_m p_{m,s}(m,x) < \left[\dfrac{s}{2}\right]$, *where* $0!! = 1$ *by definition.*

Proof. For $s = 0$ or $s = 1$, relations (2.12)-(2.14) are true. Let $s \in \mathbb{N}, s \geq 2$. Considering (2.4) it results that in (2.12) it is necessary that $\alpha_{m,s} = 0$ for s even. We prove (2.12) through induction after s. For $s = 2$, $T_{m,2}(x) = [x(1-x)]\big(\alpha_{m,2}x + \beta_{m,2}\big)m + p_{m,2}(m,x)$,where $\alpha_{m,2} = 0$, $\beta_{m,2} = 1$ and $p_{m,2}(m,x) = 0$.

Assume that $T_{m,n}(x) = [x(1-x)]^{\left[\frac{n}{2}\right]}\big(\alpha_{m,n}x + \beta_{m,n}\big)m^{\left[\frac{n}{2}\right]} + p_{m,n}(m,x)$, $grad_m p_{m,n}(m,x) < \left[\dfrac{n}{2}\right]$, $\forall\, n \in \mathbb{N}$, $n \in \{2, 3, \ldots, s\}$.

According to (1.3), from Lemma 1.1 we obtain

$$\begin{aligned} &T_{m,s+1}(x) = x(1-x)\left[T'_{m,s}(x) + msT_{m,s-1}(x)\right] = x(1-x). \\ &\left\{\left[\left[\frac{s}{2}\right][x(1-x)]^{\left[\frac{s}{2}\right]-1}(1-2x)\big(\alpha_{m,s}x + \beta_{m,s}\big) + [x(1-x)]^{\left[\frac{s}{2}\right]}\alpha_{m,s}\right]m^{\left[\frac{s}{2}\right]} + \right. \\ &+ p'_{m,s}(m,x) + [x(1-x)]^{\left[\frac{s-1}{2}\right]}\big(\alpha_{m,s-1}x + \beta_{m,s-1}\big)s\,m^{\left[\frac{s-1}{2}\right]+1} + \\ &\left. +\ ms\,p_{m,s-1}(m,x)\right\} = [x(1-x)]^{\left[\frac{s}{2}\right]}(1-2x)\big(\alpha_{m,s}x + \beta_{m,s}\big)\left[\frac{s}{2}\right]m^{\left[\frac{s}{2}\right]} + \\ &+ [x(1-x)]^{\left[\frac{s}{2}\right]+1}\alpha_{m,s}m^{\left[\frac{s}{2}\right]} + x(1-x)p'_{m,s}(m,x) + \\ &+ [x(1-x)]^{\left[\frac{s-1}{2}\right]+1}\big(\alpha_{m,s-1}x + \beta_{m,s-1}\big)s\,m^{\left[\frac{s-1}{2}\right]+1} + x(1-x)msp_{m,s-1}(m,x)\,. \end{aligned}$$

This identity proves that if s is even, which means $s = 2k$, $k \in \mathbb{N}^*$, we have

$$\begin{aligned} T_{m,2k+1}(x) = [x(1-x)]^k &\left[(1-2x)k\beta_{m,2k} + 2k\big(\alpha_{m,2k-1}\,x + \beta_{m,2k-1}\big)\right]m^k + \\ &+ x(1-x)\left[p'_{m,2k}(m,x) + 2km\,p_{m,2k}(x)\right], \end{aligned}$$

so

$$\begin{aligned} T_{m,2k+1}(x) = [x(1-x)]^{\left[\frac{2k+1}{2}\right]} &\left[\big(2k\,\alpha_{m,2k-1} - 2k\,\beta_{m,2k}\big)x + \right. \\ &\left. + k\,\beta_{m,2k} + 2k\,\beta_{m,2k-1}\right]m^{\left[\frac{2k+1}{2}\right]} + \\ &+ x(1-x)\left[p'_{m,2k}(m,x) + 2km\,p_{m,2k-1}(m,x)\right] \end{aligned} \tag{2.13}$$

and so

$$\alpha_{m,2k+1} = 2k\,\alpha_{m,2k-1} - 2k\,\beta_{m,2k}\,, \quad \forall\, k \in \mathbb{N},\ k \geq 2 \tag{2.14}$$

and

$$\beta_{m,2k+1} = k\,\beta_{m,2k} + 2k\,\beta_{m,2k-1}\,, \quad \forall\, k \in \mathbb{N},\ k \geq 2. \tag{2.15}$$

Similarly, if s is odd, which means $s = 2k+1$, $k \in \mathbb{N}^*$, we have

$$\begin{aligned} T_{m,2k+2}(x) = {} & [x(1-x)]^{k+1}(2k+1)\beta_{m,2k}m^{k+1} + \\ & + [x(1-x)]^{k}(1-2x)\big(\alpha_{m,2k+1}x + \beta_{m,2k+1}\big)km^{k} + \\ & + [x(1-x)]^{k+1}\alpha_{m,2k+1}m^{k} + x(1-x)\,\big[p'_{m,2k+1}(m,x) + \\ & +(2k+1)p_{m,2k}(m,x)\big] \end{aligned} \tag{2.16}$$

and so

$$\beta_{m,2k+2} = (2k+1)\beta_{m,2k}\,, \quad \forall\, k \in \mathbb{N}^*\,. \tag{2.17}$$

Relations (2.15) - (2.19) yield to the end of the induction.
Next, we determine $\alpha_{m,s},\ \beta_{m,s},\ s \in \mathbb{N},\ s \geq 2.$
Therefore $T_{m,2}(x) = x(1-x)m$ gives us that $\beta_{m,2} = 1$ and then from (2.19) we obtain

$$\beta_{m,2n} = (2n-1)!!, \forall n \in \mathbb{N}^*. \tag{2.18}$$

From $T_{m,3}(x) = x(1-x)(1-2x)m$, $\alpha_{m,3}(x) = -2$ and $\beta_{m,3}(x) = 1$, follow.

Now, from (2.16), (2.17) and (2.20) it results that

$$a_{m,2n+1} = -(2n)!!\sum_{k=1}^{n}\frac{(2k-1)!!}{(2k-2)!!} \tag{2.19}$$

and

$$\beta_{m,2n+1} = \frac{1}{2}(2n)!!\sum_{k=1}^{n}\frac{(2k-1)!!}{(2k-2)!!}, \tag{2.20}$$

for all $n \in \mathbb{N}^*$.
Considering that $\alpha_{m,2n} = 0$, $\forall\ n \in \mathbb{N}^*$, (2.20) - (2.22) give (2.13) and (2.14) as outcomes.

Because the numbers $\alpha_{m,s}$ and $\beta_{m,s}$ from Theorem 2.2 does not depend on m, Theorem 2.2 is reforming through Corollary 2.1. □

The results contained in Corollary 2.1, Corollary 2.2 and Corollary 2.3 are known (see [2] or [3]).

Corollary 2.1. *Let $m \in \mathbb{N}^*$ and $s \in \mathbb{N}$. Then*

$$T_{m,s}(x) = [x\,(1-x)]^{\left[\frac{s}{2}\right]}(\alpha_s x + \beta_s)\,m^{\left[\frac{s}{2}\right]} + p_{m,s}(m,x), \tag{2.21}$$

$$\alpha_s = \begin{cases} 0, & \textit{if}\ \ s\ \ \textit{is even or}\ s = 1 \\ -(s-1)!!\sum\limits_{k=1}^{\left[\frac{s}{2}\right]}\dfrac{(2k-1)!!}{(2k-2)!!}, & \textit{if}\ \ s\ \textit{is odd},\ s \geq 3, \end{cases} \tag{2.22}$$

$$\beta_s = \begin{cases} 1, & \textit{if } s = 0 \\ 0, & \textit{if } s = 1 \\ (s-1)!!, & \textit{if } s \textit{ is even}, \ s \geq 2 \\ \frac{1}{2}(s-1)!! \sum\limits_{k=1}^{\left[\frac{s}{2}\right]} \dfrac{(2k-1)!!}{(2k-2)!!}, & \textit{if } s \textit{ is odd}, \ s \geq 3 \end{cases} \tag{2.23}$$

and $gr_m p_{m,s}(m,x) < \left[\frac{s}{2}\right]$.

Corollary 2.2. *If* $s \in \mathbb{N}$, *then*

$$\lim_{m\to\infty} \frac{T_{m,s}(x)}{m^{\left[\frac{s}{2}\right]}} = [x(1-x)]^{\left[\frac{s}{2}\right]}\left(\alpha_s x + \beta_s\right), \quad \forall\, x \in [0,1]. \tag{2.24}$$

Proof. Demonstration follows from Corollary 2.1. □

Corollary 2.3. *Let* $s \in \mathbb{N}$ *be an even number. Then*

$$\lim_{m\to\infty} \frac{T_{m,s}(x)}{m^{\frac{s}{2}}} = \begin{cases} 1, & \textit{if } s = 0 \\ [x(1-x)]^{\frac{s}{2}}(s-1)!!, & \textit{if } s \geq 2 \end{cases} \tag{2.25}$$

for all $x \in [0,1]$.

Proof. Again, the proof is a consequence of Corollary 2.2. □

References

[1] Agratini O., *Approximation by linear operators*, Presa Universitară Clujeană, Cluj-Napoca, 2000 (in Romanian)

[2] Bernstein S., *Complemént à l'article de E.Voronovskaja*, C. R. Acad. Sci. URSS (1932), 86-92

[3] Lorentz G. G., *Bernstein polynomials*, University of Toronto Press, Toronto, 1953

[4] Pop O. T., *The generalization of Voronovskaja's theorem for a class of linear and positive operators*, Revue. Anal. Num. Theor. Approx., **34** (2005), no. 1, 79-91

[5] Pop O. T., *About the generalization of Voronovskaja's theorem*, Annals of University of Craiova, Math. Comp. Sci. Ser., **31**, 2004

[6] Stancu D. D., Coman, Gh., Agratini, O., Trîmbiţaş, R., *Numerical analysis and approximation theory*, **I**, Presa Universitară Clujeană, Cluj-Napoca, 2001 (in Romanian)

[7] Voronovskaja E., *Détermination de la forme asymtotique d'approximation des fractions par les polynomials de M.Bernstein*, C. R. Acad. Sci. URSS (1932), 79-85

National College "Mihai Eminescu"
5 Mihai Eminescu Street
Satu Mare 440014, Romania

Vest University "Vasile Goldiş" of Arad
Branch of Satu Mare
26 Mihai Viteazul Street
Satu Mare 440030, Romania
E-mail address: ovidiutiberiu@yahoo.com