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Extensions of some Ceva type theorems in polygons

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ABSTRACT. A complete characterization of the concurrence conditions of some straight lines that pass through the vertices of a triangle is given by Ceva Theorem. The problems connected with the concurrence of some lines in polygons, polyhedrons or simplexes lead to the idea of the extension of this result in other concurrence theorems.

INTRODUCTION

The theorems and problems referring to the concurrence of straight lines in the triangle, rely essentially on the Theorem of Ceva, which gives a complete characterization for the concurrence of some lines that pass through the vertices of the triangle.

In the geometry of the triangle, the statement referring to the concurrence of some straight lines, although diverse, can be considered classical problems. The extension of some theorems and problems from triangles to other polygons open a new field for the generalization of some known results. The purpose of this paper is to present an extension of the most usual theorems, the general setting of extension being the convex polygons with an odd number of sides.

1. PRELIMINARY NOTIONS AND RESULTS

The most known concurrence theorems in the triangle concern the concurrence of some important lines: the concurrence of the medians, the concurrence of the perpendicular bisectors of sides, the concurrence of the bisectors, the concurrence of the heights and Ceva's Theorem. We remind the statements of these theorem:

Theorem 1.1. *In a triangle the perpendicular bisectors of sides are concurrent (in the circumcenter of triangle).*

Theorem 1.2. *In a triangle the bisectors are concurrent (in the incenter of triangle).*

Theorem 1.3. *In a triangle the medians are concurrent (in the centroid of triangle).*

Theorem 1.4. *In a triangle the heights are concurrent (in the orthocenter of triangle).*

Theorem 1.5. (Ceva's Theorem) *In a triangle ABC the cevians AA', BB', CC' are concurrent if and only if it is satisfied the relation:*

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1.$$

The results synthesized in this paper are extensions of the previous theorems. It is easy to see that any attempts to define the notion of median, height, cevian in

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a quadrilateral or in polygons with an even number of sides is unnatural. These notions can be extended naturally for polygons with an odd number of sides.

Let $\mathcal{P} = [A_1A_2 \dots A_{2n}A_{2n+1}]$ be a convex polygon.

Definition 1.1. Is called median of $\mathcal{P}$ any straight line passing through a vertex of a polygon $\mathcal{P}$ and the midpoint of opposite side. (The opposite side of the vertex A_k is the side $[A_{k+n}A_{k+n+1}]$, the numbering being made with the convention $A_{2n+1+p} = A_p$.)

Definition 1.2. It is called height in the polygon $\mathcal{P}$ any straight line that passes through a vertex of the polygon and is perpendicular on the support line of the opposite side.

Definition 1.3.1. It is called bisector of the polygon $\mathcal{P}$ any bisector line of the angle of the polygon $(\widehat{A_1A_2}A_3, \widehat{A_2A_3}A_4, \dots, A_{2n}\widehat{_{+1}A_1}A_2)$.

Definition 1.3.2. It is called "interior bisector" in the polygon $\mathcal{P}$ the bisector lines of the angles $A_{n+k}\widehat{A_kA}_{n+k+1}$, where A_k is a vertex of the polygon and $[A_{n+k}A_{n+k+1}]$ is its opposite side.

Definition 1.4. It is called perpendicular bisector in the polygon $\mathcal{P}$ any straight line that passes through the midpoint of a side and is perpendicular of it.

Definition 1.5. It is called cevian in the polygon $\mathcal{P}$ a straight line that links a vertex of the polygon with a point of the opposite side.

2. The concurrence of bisectors and perpendicular bisectors

Since the notion of inscribed and circumscribed polygon are well-known, we confine ourselves to remind the natural extensions of Theorems 1.1 and 1.2 for convex polygons with an arbitrary number of sides.

Theorem 2.1. *In a convex polygon the bisector lines are concurrent if and only if the polygon is circumscrible. (The intersection point of the bisector lines is the center of the inscribed circle in polygon.)*

Theorem 2.2. *In a convex polygon the perpendicular bisectors of all sides are concurrent if and only if the polygon is inscrible. (The intersection point of the perpendicular bisectors is center of the circumscribed circle of polygon.)*

3. The concurrence of the medians

One can easily construct examples of polygons with an odd number $2n + 1 \geq 5$ of sides, in which any three medians are not concurrent (a simple construction can be given in a pentagon). This means that Theorem 1.1 cannot be extended in the given form. That's why we give an equivalent statement for the concurrence of medians.

Theorem 1.1.(3) *If two of the medians of a triangle have a common point G then the third median passes through G.*

The natural extension of this theorem to a polygon $\mathcal{P}$ with an odd number of sides is now possible and we will show that it is true.

Theorem 3.1. *If in a polygon $\mathcal{P} = [A_1A_2 \dots A_{2n}A_{2n+1}]$, $2n$ of the medians are concurrent at G, then the other median also passes through G.*

Proof. We choose a system of coordinates with the origin at the point G, intersection of the medians from $A_1, A_2, \dots, A_{2n}$ and we denote $\overline{a_1}, \overline{a_2}, \dots, \overline{a_{2n}}, \overline{a_{2n+1}}$

the vectors $\overline{GA_1}$, $\overline{GA_2}, \ldots, \overline{GA_{2n}}$, $\overline{GA_{2n+1}}$ and by $B_1, B_2, \ldots, B_{2n}, B_{2n+1}$ the midpoints of the opposite sides. The condition that the points A_k, G, B_k are collinear is equivalent to:

$$\overline{a_k} \times \frac{1}{2}(\overline{a_{n+k}} + \overline{a_{n+k+1}}) = \overline{0}, \quad k = \overline{1, 2n}.$$

We obtain the system:

$$\begin{array}{ll}
(1): & \overline{a_1} \times (\overline{a_{n+1}} + \overline{a_{n+2}}) = \overline{0} \\
(2): & \overline{a_2} \times (\overline{a_{n+2}} + \overline{a_{n+3}}) = \overline{0} \\
 & \ldots \\
(n): & \overline{a_n} \times (\overline{a_{2n}} + \overline{a_{2n+1}}) = \overline{0} \\
(n+1): & \overline{a_{n+1}} \times (\overline{a_{2n+1}} + \overline{a_1}) = \overline{0} \\
(n+2): & \overline{a_{n+2}} \times (\overline{a_1} + \overline{a_2}) = \overline{0} \\
 & \ldots \\
(2n): & \overline{a_{2n}} \times (\overline{a_{n-1}} + \overline{a_n}) = \overline{0}
\end{array}$$

Adding these equalities we notice that the product $\overline{a_1} \times \overline{a_{n+1}}$ from (1) reduced with $\overline{a_{n+1}} \times \overline{a_1}$ from $(n+1)$, $\overline{a_1} \times \overline{a_{n+2}}$ from (1) reduced with $\overline{a_{n+2}} \times \overline{a_1}$ from $(n+2), \ldots$. It rest only the products where $\overline{a_{2n+1}}$ appears (just on the second position in (n) and first position in $(n+1)$), and it follows that $\overline{a_n} \times \overline{a_{2n+1}} + \overline{a_{n+1}} \times \overline{a_{2n+1}} = \overline{0} \Leftrightarrow \overline{a_{2n+1}} \times (\overline{a_n} + \overline{a_{n+1}}) = \overline{0}$, so the points A_{2n+1}, G, B_{2n+1} are collinear.

Remark 3.1. Following the analogy between the vectorial plane geometry and the geometry of the complex plane, the theorem can be also proved using complex numbers but the proof is not simple.

Remark 3.2. A variant of this problem was given to the Balcanic Olympiad in 1998 for a pentagon, but the proof is difficult for those which do not know the general theorem about cevians (Theorem 6.1). The solution of this problem using complex numbers can be found in [4].

4. THE CONCURRENCE OF THE HEIGHTS

One can easily imagine an example of polygon (pentagon) in which any three heights are not concurrent. To extend the Theorem 1.4 we formulate the statement in the form: **Theorem 1.2.(4)** *If two of the heights of a triangle intersect at H, then the third height passes through H.*

With this statement, the theorem is extended in the form:

Theorem 4.1. *If in the polygon $\mathcal{P} = [A_1A_2 \ldots A_{2n}A_{2n+1}]$, $2n$ of the heights are concurrent at H, then the other height also passes through H.*

Proof. We choose in the polygons plane a system of axes with the origin at H and we denote the vectors $\overline{HA_1} = \overline{a_1}$, $\overline{HA_2} = \overline{a_2}, \ldots, \overline{HA_{2n}} = \overline{a_{2n}}$, $\overline{HA_{2n+1}} = \overline{a_{2n+1}}$. The condition $HA_1 \perp A_{n+1}A_{n+2}$ is written $\overline{a_1} \cdot (\overline{a_{n+2}} - \overline{a_{n+1}}) = 0$ or $\overline{a_1} \cdot \overline{a_{n+1}} = \overline{a_1} \cdot \overline{a_{n+2}}$. In the same manner, from the conditions $HA_2 \perp A_{n+2}A_{n+3}, \ldots, HA_{2n} \perp$

$A_{n-1}A_n$ we obtain the system of relations:

$$\begin{array}{ll}(1): & \overline{a_1}\cdot\overline{a_{n+1}}=\overline{a_1}\cdot\overline{a_{n+2}}\\ (2): & \overline{a_2}\cdot\overline{a_{n+2}}=\overline{a_2}\cdot\overline{a_{n+3}}\\ & \dots\\ (n): & \overline{a_n}\cdot\overline{a_{2n}}=\overline{a_n}\cdot\overline{a_{2n+1}}\\ (n+1): & \overline{a_{n+1}}\cdot\overline{a_{2n+1}}=\overline{a_{n+1}}\cdot\overline{a_1}\\ (n+2): & \overline{a_{n+2}}\cdot\overline{a_1}=\overline{a_{n+2}}\cdot\overline{a_2}\\ & \dots\\ (2n): & \overline{a_{2n}}\cdot\overline{a_{n-1}}=\overline{a_{2n}}\cdot\overline{a_n}\end{array}$$

If we add all the equalities, the left-hand terms of the equalities $(1),(2),\dots,(n)$ reduced with the right-hand terms of the relations $(n+1),(n+2),\dots,(2n)$ and the left-hand terms of the equalities $(n+2),\dots,(2n)$ reduced with the right-hand ones of the equalities $(1),\dots,(n-1)$. On the right remains the term $\overline{a_{n+1}}\cdot\overline{a_{2n+1}}$ from $(n+1)$ and on the left remains the term $\overline{a_n}\cdot\overline{a_{2n+1}}$ from (n), so $\overline{a_n}\cdot\overline{a_{2n+1}}=\overline{a_{n+1}}\cdot\overline{a_{2n+1}}$ or $\overline{a_{2n+1}}\cdot(\overline{a_{n+1}}-\overline{a_n})=0$, which means that $HA_{2n+1}\perp A_nA_{n+1}$.

Remark 4.2. The problem was given at the Olympiad in Russia and can be found in [1], problem R114 (R. Jenodarov).

For the polygons with even numbers of sides, the notion of "height" and a similar theorem with Theorem 4.1 was given by Gh. D. Simionescu [5].

Definition 4.3. If $\mathcal{P}=[A_1A_2\dots A_{2n-1}A_{2n}]$ is a polygon with even number sides, we call "height" a straight line which passes through the midpoint of a side $[A_kA_{k+1}]$ and is perpendicular of the opposite side $[A_{n+k}A_{n+k+1}]$.

Theorem 4.4. *If $(2n-1)$ "heights" of a polygon with $2n$ sides are concurrent, then all the $2n$ "heights" are concurrent.*

Proof. First we show that if M_k is the midpoint of side $[A_{k+1}A_{k+2}]$ and H is an arbitrary point in plane, then the following relation holds: (4)

$$\overline{HM_1}\cdot\overline{A_{n+1}A_{n+2}}+\overline{HM_2}\cdot\overline{A_{n+2}A_{n+3}}+\cdots+\overline{HA_{2n-1}}\cdot\overline{A_{n-1}A_n}+\overline{HA_{2n}}\cdot\overline{A_nA_{n+1}}=0.$$

If we denote $\overline{HA_k}=\overline{a}_k$, $k=1,2,\dots,2n$ the relation (4) gives:

$$\frac{1}{2}\sum_{k=1}^{n}(\overline{a}_k+\overline{a}_{k+1})(\overline{a}_{n+k+1}-\overline{a}_{n+k})=0 \Leftrightarrow$$

$$\sum_{k=1}^{n}\overline{a}_k\cdot\overline{a}_{k+n+1}-\sum_{k=1}^{n}\overline{a}_k\cdot\overline{a}_{k+n}+\sum_{k=1}^{n}\overline{a}_{k+1}\cdot\overline{a}_{k+n+1}-\sum_{k=1}^{n}\overline{a}_{k+1}\cdot\overline{a}_{k+n}=0$$

(the first sum is equal to the fourth, and the second to the third, because of $\overline{a}_k=\overline{a}_{k+2n}$).

If we denote by H the intersection point of the "heights" from $M_1,M_2,\dots,M_{2n-1}$, then the first $(2n-1)$ terms from (4) are equal to zero, so the last term $\overline{HM_{2n}}\cdot\overline{A_nA_{n+1}}=0$ or $HM_{2n}\perp A_nA_{n+1}$. Then the straight line HM_{2n} is a "height".

5. The concurrence of the "interior bisector" lines

Theorem 5.1. *If in the polygon $\mathcal{P}=[A_1A_2\dots A_{2n}A_{2n+1}]$, $2n$ of the "interior bisector" lines are concurrent, then all the "interior bisector" lines are concurrent.*

Proof. Let I be the intersection of "interior bisector" lines of the angles with the vertices $A_1, A_2, \ldots, A_{2n}$. We denote by $\Delta_1 = A_1A_{n+1}$, $\Delta_2 = A_2A_{n+2}, \ldots,$ $\Delta_{2n} = A_{2n}A_{n-1}$, $\Delta_{2n+1} = A_{2n+1}A_1$, the "biggest" diagonals of the polygons and by $d_1, d_2, \ldots, d_{2n}, d_{2n+1}$ the distance from the point I to the diagonals $\Delta_1, \Delta_2, \ldots, \Delta_{2n}, \Delta_{2n+1}$. Since the point I belongs to the bisector lines of the angle $\widehat{A_{n+1}A_1A_{n+2}}$ we have $d_1 = d_{n+2}$ and writing the conditions that the first $2n$ "interior bisector" lines pass through I, we obtain the relations:

$$\begin{array}{ll}
(1): & d_1 = d_{n+2} \\
(2): & d_2 = d_{n+3} \\
 & \ldots \\
(n-1): & d_{n-1} = d_{2n} \\
(n): & d_n = d_{2n+1} \\
(n+1): & d_{n+1} = d_1 \\
(n+2): & d_{n+2} = d_2 \\
 & \ldots \\
(2n): & d_{2n} = d_n
\end{array}$$

If we add all the equalities we obtain $d_{n+1} = d_{2n+1}$, hence the point I is on the bisector lines of the angle $\widehat{A_nA_{2n+1}A_{n+1}}$, the last "interior bisector" line.

Remark 5.1. If the polygon $\mathcal{P}$ has the "interior bisector" lines concurrent at I, then the "biggest" diagonals determine a polygon circumscribed to a circle with the center at I.

6. The concurrence of the cevians

Let $\mathcal{P} = [A_1A_2 \ldots A_{2n}A_{2n+1}]$ be a convex polygon and the points $B_1 \in [A_{n+1}A_{n+2}]$, $B_2 \in [A_{n+2}A_{n+3}], \ldots,$ $B_{2n+1} \in [A_nA_{n+1}]$ on the sides opposite to the vertices $A_1, A_2, \ldots, A_{2n+1}$.

Theorem 6.1. (Generalization of Ceva's theorem) *It the cevians A_1B_1, $A_2B_2, \ldots,$ $A_{2n}B_{2n}$, $A_{2n+1}B_{2n+1}$ are concurrent, then the following equality is satisfied:*

$$\frac{A_{n+1}B_1}{B_1A_{n+2}} \cdot \frac{A_{n+2}B_2}{B_2A_{n+3}} \cdots \frac{A_{n-1}B_{2n}}{B_{2n}A_n} \cdot \frac{A_nB_{2n+1}}{B_{2n+1}A_{n+1}} = 1.$$

Proof. Let I be the intersection point of the cevians. We denote the angles $\alpha_1 = \widehat{A_{n+1}IB_1}$, $\beta_1 = \widehat{B_1IA_{n+2}}$, $\alpha_2 = \widehat{A_{n+2}IB_2}$, $\beta_2 = \widehat{B_2IA_{n+3}}, \ldots,$ $\alpha_{2n+1} = \widehat{A_{n+1}IB_{2n+1}}$, $\beta_{2n+1} = \widehat{B_{2n+1}IA_{n+2}}$.

Obviously, we have the equality of the angles that are opposite

$$\alpha_1 = \beta_{n+1}, \alpha_2 = \beta_{n+2}, \ldots, \alpha_{n+1} = \beta_{2n+1}, \alpha_{n+2} = \beta_1, \ldots, \alpha_{2n+1} = \beta_n.$$

We express the fractions from the statement using areas:

$$\frac{A_{n+1}B_1}{B_1A_{n+2}} = \frac{\sigma(A_{n+1}IB_1)}{\sigma(B_1IA_{n+2})} = \frac{IA_{n+1} \cdot IB_1 \cdot \sin\alpha_1}{IB_1 \cdot IA_{n+2} \cdot \sin\beta_1} = \frac{IA_{n+1}}{IA_{n+2}} \cdot \frac{\sin\alpha_1}{\sin\beta_1}.$$

In the same way we express the other rapports and we obtain:

$$\frac{A_{n+1}B_1}{B_1A_{n+2}} \cdot \frac{A_{n+2}B_2}{B_2A_{n+3}} \cdots \frac{A_{n-1}B_{2n}}{B_{2n}A_n} \cdot \frac{A_nB_{2n+1}}{B_{2n+1}A_{n+1}} =$$

$$= \left(\frac{IA_{n+1}}{IA_{n+2}} \cdot \frac{IA_{n+2}}{IA_{n+3}} \cdots \frac{IA_{n-1}}{IA_n} \cdot \frac{IA_n}{IA_{n+1}}\right)\left(\frac{\sin\alpha_1}{\sin\beta_1} \cdot \frac{\sin\alpha_2}{\sin\beta_2} \cdots \frac{\sin\alpha_{2n}}{\sin\beta_{2n}} \cdot \frac{\sin\alpha_{2n+1}}{\sin\beta_{2n+1}}\right) = 1,$$

since in the first bracket at numerator and denominator appear all the length $IA_1, IA_2, \dots, IA_{2n}, IA_{2n+1}$, and in the second bracket the angles $(\beta_1, \beta_2, \dots, \beta_{2n+1})$ represent a circular permutation of the angles $(\alpha_1, \alpha_2, \dots, \alpha_{2n+1})$.

Remark 6.2. Theorem 6.1 can be found in a book for the preparation of the Bulgarian olympic team.

7. The concurrence of the heights in the tetrahedron

We noticed that in a quadrilateral the notion of height is not natural, but if we take the four vertices not in the same plane we obtain a tetrahedron in which the heights are natural defined. It is known that generally the heights of a tetrahedron are not concurrent (not even two by two), the tetrahedron with the property that the heights are concurrent are particular tetrahedron (called orthocentric) and they have many remarkable properties.

In the spirit of the previous theorem about polygons we searched something similar in tetrahedrons and we obtain:

Theorem 7.1. *If in a tetrahedron three of the heights are concurrent then all the heights are concurrent.*

Proof. Let $ABCD$ be a tetrahedron and H the intersection of the heights from A, B and C. We denote the vectors $\overline{HA} = \overline{a}$, $\overline{HB} = \overline{b}$, $\overline{HC} = \overline{c}$ and $\overline{HD} = \overline{d}$. From $HA \perp (BCD)$ we have $\overline{a} \cdot \overline{BC} = 0$, $\overline{a} \cdot \overline{BD} = 0$ or $\overline{a} \cdot (\overline{c} - \overline{b}) = 0$ and $\overline{a} \cdot (\overline{d} - \overline{b}) = 0$, so $\overline{a} \cdot \overline{c} = \overline{a} \cdot \overline{b} = \overline{a} \cdot \overline{d}$. In the same way, from $HB \perp (ACD)$ and $HC \perp (ABD)$ follows $\overline{b} \cdot \overline{a} = \overline{b} \cdot \overline{c} = \overline{b} \cdot \overline{d}$ and $\overline{c} \cdot \overline{a} = \overline{c} \cdot \overline{b} = \overline{c} \cdot \overline{d}$. In conclusion $\overline{a} \cdot \overline{b} = \overline{a} \cdot \overline{c} = \overline{a} \cdot \overline{d} = \overline{b} \cdot \overline{c} = \overline{b} \cdot \overline{d} = \overline{c} \cdot \overline{d}$ from which we keep the relations $\overline{d} \cdot \overline{a} = \overline{d} \cdot \overline{b} = \overline{d} \cdot \overline{c}$. Writing them in the form: $\overline{d} \cdot (\overline{b} - \overline{a}) = 0$ and $\overline{d} \cdot (\overline{c} - \overline{a}) = 0$ it follows $\overline{d} \perp AB$, $\overline{d} \perp AC$, so $HD \perp (ABC)$.

Another extension of Theorem 1.4 in three dimensions was given by Simionescu [5].

Theorem 7.2. *If in a pyramid with $(2n+1)$ faces, $2n$ edges are orthogonal on the opposite edges of the bases, then the last edge is perpendicular on the opposite side.*

Proof. One can prove easily that if $P = [A_1A_2 \dots A_{2n}A_{2n+1}]$ is the base and S is the vertex of the pyramid then:

$$\overline{SA_1} \cdot \overline{A_{n+1}A_{n+2}} + \overline{SA_2} \cdot \overline{A_{n+2}A_{n+3}} + \cdots + \overline{SA_{2n}} \cdot \overline{A_{n-1}A_n} + \overline{SA_{2n+1}} \cdot \overline{A_nA_{n+1}} = 0.$$

Since the first $2n$ terms of the above relation are zero it follows that $SA_{2n+1} \perp A_nA_{n+1}$.

References

[1] Agahanov N., Podlipsky O., *Russian mathematical olympiads*, Editura Gil, Zalău, 2004 (in Romanian)
[2] Miculiţa M., *Introduction to the geometry of the tetrahedron*, Editura Minied, Iaşi, 1994 (in Romanian)
[3] Pop V., Lupşor V., (colectiv), *Mathematics for excellent students' groups (grade X textbook)*, Editura Dacia Educaţional, 2003 (in Romanian)
[4] Pop V., Lupşor V., (colectiv), *Mathematics for excellent students' groups (grade X textbook)*, Editura Dacia Educaţional, 2004 (in Romanian)
[5] Simionescu Gh. D., *Notions of vectorial algebra and applications in geometry*, Editura Tehnică, Bucureşti, 1982 (in Romanian)

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