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On a result of D. D. Stancu

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ABSTRACT. We will present two results concerning some approximation properties of the bivariate operator of D.D. Stancu, using a different technique. The article presents a teaching method of calculation.

1. INTRODUCTION

For $\mathbb{R}^I = \{f; f : I \to \mathbb{R}, I = [0,1]\}$, in [5] D.D. Stancu has introduced the operator $S_m^{\alpha,\beta}$, defined for any function $f \in \mathbb{R}^I$, by

$$\left(S_m^{\alpha,\beta} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.1}$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \quad \text{for any} \quad 0 \le \alpha \le \beta. \tag{1.2}$$

Assume that $f \in \mathbb{R}^{I^2}$, $0 \le \alpha_1 \le \beta_1$ and $0 \le \alpha_2 \le \beta_2$. Denote by ${}_xS_m^{(\alpha_1,\beta_1)}$, ${}_yS_n^{(\alpha_2,\beta_2)}$ the parametric extensions of $S_m^{(\alpha,\beta)}$, i.e.

$$\left({}_xS_m^{(\alpha_1,\beta_1)} f\right)(x,y) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha_1}{m+\beta_1}, y\right) \tag{1.3}$$

$$\left({}_yS_n^{(\alpha_2,\beta_2)} f\right)(x,y) = \sum_{j=0}^{n} p_{n,j}(y) f\left(x, \frac{j+\alpha_2}{n+\beta_2}\right) \tag{1.4}$$

where the fundamental polynomials $p_{n,j}(y)$ are analogues with (1.2).

Proposition 1.1. *The parametric extensions ${}_xS_m^{(\alpha_1,\beta_1)}$, ${}_yS_n^{(\alpha_2,\beta_2)}$ satisfy*

$${}_xS_m^{(\alpha_1,\beta_1)} \cdot {}_yS_n^{(\alpha_2,\beta_2)} = {}_yS_n^{(\alpha_2,\beta_2)} \cdot {}_xS_m^{(\alpha_1,\beta_1)} = S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}.$$

Their product is the bivariate operator $S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : \mathbb{R}^{I^2} \to \mathbb{R}^{I^2}$ which for any function $f \in \mathbb{R}^{I^2}$ gives the approximation

$$\left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right). \tag{1.5}$$

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Properties of the bivariate operator of Stancu type were studied by D. Bărbosu in [1], [2], [3] and other papers. Next, we will present some of them.

Theorem 1.1. *The bivariate operator of Stancu type has the following properties:* (i) *it is linear and positive;*

(ii) $\left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} e_{00}\right)(x,y) = 1;$

$\left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} e_{10}\right)(x,y) = x + \frac{\alpha_1 - x\beta_1}{m+\beta_1}\,;$

$\left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} e_{01}\right)(x,y) = y + \frac{\alpha_2 - y\beta_2}{n+\beta_2}\,;$

$\left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} e_{20}\right)(x,y) = x^2 + \frac{mx(1-x)}{(m+\beta_2)} + \frac{(\alpha_1 - x\beta_1)(2mx+\beta_1 x+\alpha_1)}{(m+\beta_1)^2}\,;$

$\left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} e_{02}\right)(x,y) = y^2 + \frac{ny(1-y)}{(n+\beta_2)^2} + \frac{(\alpha_2 - y\beta_2)(2ny+\beta_2 y+\alpha_2)}{(n+\beta_2^2)},$

where we denote by $e_{ij}(x,y) = x^i y^j$ $(i = \overline{0,2}, j = \overline{0,2})$ *the test functions.*
(iii) *if* $f \in C(I^2)$,
then the sequence $\left\{S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right\}_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$ *is uniformly convergent to* f *in* I^2.

Note that for $\alpha = \beta = 0$ the operator (1.1) is the classical Bernstein operator.

2. Main results

Theorem 2.2. (Stancu, D. D. [5]) *If* $f \in C(I^2)$, *then we have the estimation*

$$\left|f(x,y) - \left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right)(x,y)\right| \le \left(\frac{3}{2}\right)^2 \omega\left(f\,, \frac{\sqrt{m+4\beta_1}}{m+\beta_1}, \frac{\sqrt{n+4\beta_2}}{n+\beta_2}\right),$$

where ω *denotes the first order modulus of smoothness.*

We use the next result concerning the order of approximation.

Theorem 2.3. (Shisha, Mond, [3]) *Let* X *and* Y *be two compact intervals from* $\mathbb{R}$. *Consider* $L : C(X\times Y) \to C(X\times Y)$ *a linear and positive operator. Then for any* $f \in C(X\times Y)$, (x,y) *and* $\delta_1, \delta_2 > 0$ *is valid the estimation*

$$\begin{aligned}
|(f - L(f))(x,y)| &\le |f(x,y)|\cdot|1 - L(1;x,y)| \\
&+ \Bigg[L(1;x,y) + \frac{1}{\delta_1}\sqrt{L\big((x-\cdot)^2;x,y\big)L(1;x,y)} \\
&+ \frac{1}{\delta_2}\sqrt{L\big((y-*)^2;x,y\big)L(1;x,y)} \\
&+ \frac{1}{\delta_1\delta_2}\sqrt{L\big((x-\cdot)^2;x,y\big)L\big((y-*)^2;x,y\big)}\Bigg]\,\omega(\delta_1,\delta_2)\,,
\end{aligned} \tag{2.6}$$

where we denote by $x - \cdot$ *and* $y - *$ *that* x *and* y *are fixed points.*

Applying this theorem to the bivariate operator of Stancu, we obtain

$$\left|f(x,y) - S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right| \le |f(x,y)| \cdot \left|1 - S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(1;x,y)\right|$$

$$+ \Bigg[S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(1;x,y) + \frac{1}{\delta_1}\sqrt{S_{m,n}^{(\alpha_1,\beta_1,\alpha_2\beta_2)}(1;x,y) S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((x-\cdot)^2;x,y\big)}$$

$$+ \frac{1}{\delta_2}\sqrt{S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(1;x,y) S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((y-*)^2;x,y\big)}$$

$$+ \frac{1}{\delta_1\delta_2}\sqrt{S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((x-\cdot)^2;x,y\big) S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((y-*)^2;x,y\big)}\Bigg] \omega(\delta_1\delta_2)\,. \quad (2.7)$$

Proof. Using the Theorem 1.1 we can write

$$S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((x-\cdot)^2;x,y\big) = \frac{mx(1-x)}{(m+\beta_1)^2} + \frac{(\alpha_1-\beta_1 x)^2}{(m+\beta_1)^2}$$

$$S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((y-*)^2;x,y\big) = \frac{ny(1-y)}{(n+\beta_2)^2} + \frac{(\alpha_2-\beta_2 y)^2}{(n+\beta_2)^2}.$$

Hence (2.7) is equivalent to the following

$$\left|f(x,y) - S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right| \le \Bigg[1 + \frac{1}{\delta_1}\sqrt{\frac{mx(1-x)}{(m+\beta_1)^2} + \frac{(\alpha_1-\beta_1 x)^2}{(m+\beta_1)^2}}$$

$$+ \frac{1}{\delta_2}\sqrt{\frac{ny(1-y)}{(n+\beta_2)^2} + \frac{(\alpha_2-\beta_2 y)^2}{(n+\beta_2)^2}}$$

$$+\frac{1}{\delta_1\delta_2}\sqrt{\left(\frac{mx(1-x)}{(m+\beta_1)^2} + \frac{(\alpha_1-\beta_1 x)^2}{(m+\beta_1)^2}\right)\left(\frac{ny(1-y)}{(n+\beta_2)^2} + \frac{(\alpha_2-\beta_2 y)^2}{(n+\beta_2)^2}\right)}\Bigg]\omega(\delta_1,\delta_2)$$

$$\le \Bigg[1 + \frac{1}{\delta_1(m+\beta_1)}\sqrt{\frac{m}{4}+\beta} + \frac{1}{\delta_2(n+\beta_2)}\sqrt{\frac{n}{4}+\beta_2}$$

$$+\frac{1}{\delta_1\delta_2(m+\beta_1)(n+\beta_2)}\sqrt{\left(\frac{m}{4}+\beta_1\right)\left(\frac{n}{4}+\beta_2\right)}\Bigg]\omega(\delta_1,\delta_2)\,. \quad (2.8)$$

In the inequality (2.3) we put $\delta_1 = \dfrac{\sqrt{m+4\beta_1}}{m+\beta_1}$ and $\delta_2 = \dfrac{\sqrt{n+4\beta_2}}{n+\beta_2}$ and then we obtain the desired inequality. □

Lemma 2.1. *Considering that δ is a positive number and $0 \le \alpha \le \beta$, we can write*

$$\lim_{m\to\infty}(m+\beta)\sum_{\left|\frac{k+\alpha}{m+\beta}-x\right|\ge\delta} p_{m,n}(x)\left(\frac{k+\alpha}{m+\beta}-x\right)^2 = 0\,,$$

for any $x \in [0,1]$.

Proof. From (1.5), we obtain the inequality

$$(m+\beta) \sum_{\left|\frac{k+\alpha}{m+\beta}-x\right|\geq\delta} p_{m,n}(x) \left(\frac{k+\alpha}{m+\beta}-x\right)^2 \leq \frac{(m+\beta)}{\delta^2} S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((x-\cdot)^4; x,y\big)\,. \tag{2.9}$$

A simple computation yields

$$\begin{aligned} S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\phi^3)(x,y) &= \frac{1}{(m+\beta)^3}\big[m(m-1)(m-2)x^3 + 3m(m-1)x^2 \\ &+ mx + 3\alpha mx + 3\alpha x^2 m(m-1) + 3m\alpha^2 x + \alpha^3\big], \end{aligned}$$

$$\begin{aligned} S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\phi^4)(x,y) &= \frac{1}{(m+\beta)^4}\big[m(m-1)(m-2)(m-3)x^4 \\ &+ m(m-1)(m-2)(6+4\alpha)x^3 \\ &+ m(m-1)(6\alpha^2+12\alpha+7)x^2 \\ &+ m(4\alpha^3+6\alpha^2+4\alpha+1)x + \alpha^3\big]. \end{aligned}$$

Hence

$$S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((x-\cdot)^4; x,y\big) = \frac{Q(m,x,\alpha,\beta)}{(m+\beta)^4}\,, \tag{2.10}$$

where

$$\begin{aligned} Q(m,x,\alpha,\beta) &= x^4\big[3m^2 - 8m\beta - 6m - 6m\beta^2 + \beta^4\big] \\ &+ x^3\big[-6m^2 + 8m\alpha + 12m\beta + 12m\alpha\beta + 6m\beta^2 + 12m - 4\alpha\beta^2\big] \\ &+ x^2\big[3m^2 - 6m\alpha^2 - 12m\alpha - 7m + 6\alpha^2\beta^2 - 12m\alpha\beta - 4m\beta\big] \\ &+ x\big[6m\alpha^2 + 4m\alpha + m - 3\alpha^3\beta\big] + \alpha^3\,. \end{aligned} \tag{2.11}$$

From (2.4) - (2.6) it follows that

$$(m+\beta) \sum_{\left|\frac{k+\alpha}{m+\beta}-x\right|\geq\delta} p_{m,n}(x) \left(\frac{k+\alpha}{m+\beta}-x\right)^2 \leq \frac{Q(m,x,\alpha,\beta)}{\delta^2(m+\beta)^3}\,.$$

The proof of lemma is complete. □

Theorem 2.4. *If $(x,y) \in I^2$ and $f \in C(I^2)$ satisfy the properties*
(i) *f has continuous partial derivatives;*
(ii) *f has second partial derivatives,*
then

$$\begin{aligned} &\min\{m+\beta_1, n+\beta_2\}\big[S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f)(x,y) - f(x,y)\big] \\ &\leq (\alpha_1 - x\beta_1)\frac{\partial f}{\partial x}(x,y) + (\alpha_2 - y\beta_2)\frac{\partial f}{\partial y}(x,y) \\ &+ \frac{x(1-x)}{2}\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{y(1-y)}{2}\frac{\partial^2 f}{\partial y^2}(x,y)\,, \end{aligned} \tag{2.12}$$

with equality if $m+\beta_1 = n+\beta_2$.

Proof. Using the corresponding Taylor series, we get

$$f(s,t) = f(x,y) + \frac{1}{1!}\left[(s-x)\frac{\partial f}{\partial x}(x,y) + (t-y)\frac{\partial f}{\partial y}(x,y)\right]$$
$$+ \frac{1}{2!}\left[(s-x)^2\frac{\partial^2 f}{\partial x^2}(x,y) + 2(s-x)(t-y)\frac{\partial^2 f}{\partial x \partial y}(x,y) + (t-y)^2\frac{\partial^2 f}{\partial y^2}(x,y)\right]$$
$$+ (s-x)^2\mu_1(s-x) + (s-x)(t-y)\mu_2(s-x, t-y) + (t-y)^2\mu_3(t-y)\,, \quad (2.13)$$

where the mappings μ_1, μ_2, μ_3 are bounded and

$$\lim_{h\to 0}\mu_1(h) = 0, \qquad \lim_{h_1,h_2\to 0}\mu_2(h_1,h_2) = 0, \qquad \lim_{h\to 0}\mu_3(h) = 0\,.$$

In (2.8) we take $s = \dfrac{k+\alpha_1}{m+\beta_1}$ and $t = \dfrac{j+\alpha_2}{n+\beta_2}$, multiply by $p_{m,k}(x)p_{n,j}(y)$ and we obtain

$$\left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}f\right)(x,y) - f(x,y) = \frac{\partial f}{\partial x}(x,y)S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((\cdot - x);x,y\big)$$
$$+ \frac{\partial f}{\partial y}(x,y)S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((* - y);x,y\big)$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,y)S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((\cdot - x)^2;x,y\big)$$
$$+ \frac{\partial^2 f}{\partial x \partial y}(x,y)S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((\cdot - x)(* - y);x,y\big)$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x,y)S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((* - y)^2;x,y\big) + (R_{m,n}f)(x,y)\,,$$

where

$$(R_{m,n}f)(x,y) = \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+\alpha_1}{m+\beta_1} - x\right)^2 \mu_1\left(\frac{k+\alpha_1}{m+\beta_1} - x\right)$$
$$+ \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)\left(\frac{k+\alpha_1}{m+\beta_1} - x\right)\left(\frac{j+\alpha_2}{n+\beta_2} - y\right)$$
$$\cdot\, \mu_2\left(\frac{k+\alpha_1}{m+\beta_1} - x,\ \frac{j+\alpha_2}{n+\beta_2} - y\right)$$
$$+ \sum_{j=0}^{n} p_{n,j}(y)\left(\frac{j+\alpha_2}{n+\beta_2} - y\right)^2 \mu_3\left(\frac{j+\alpha_2}{n+\beta_2} - y\right)\,. \quad (2.14)$$

Since

$$\begin{aligned}
S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((\cdot - x);x,y\big) &= \frac{\alpha_1 - x\beta_1}{m+\beta_1},\\
S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((* - y);x,y\big) &= \frac{\alpha_2 - y\beta_2}{n+\beta_2},\\
S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((\cdot - x)^2;x,y\big) &= \frac{mx(1-x)}{(m+\beta_1)^2} + \frac{(\alpha_1 - x\beta_1)^2}{(m+\beta_1)^2},\\
S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((* - y)^2;x,y\big) &= \frac{ny(1-y)}{(n+\beta_2)^2} + \frac{(\alpha_2 - y\beta_2)^2}{(n+\beta_2)^2},\\
S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\big((\cdot - x)(* - y);x,y\big) &= \frac{\alpha_1 - x\beta_1}{m+\beta_1}\cdot\frac{\alpha_2 - y\beta_2}{n+\beta_2},
\end{aligned}$$

we get

$$\begin{aligned}
&\left(S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right)(x,y) - f(x,y) \le \frac{\partial f}{\partial x}(x,y)\frac{\alpha_1 - x\beta_1}{m+\beta_1} + \frac{\partial f}{\partial y}(x,y)\frac{\alpha_2 - y\beta_2}{n+\beta_2}\\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,y)\left[\frac{x(1-x)}{m+\beta_1} + \frac{(\alpha_1 - x\beta_1)^2}{(m+\beta_1)^2}\right] + \frac{\partial^2 f}{\partial x \partial y}(x,y)\,\frac{\alpha_1 - x\beta_1}{m+\beta_1}\cdot\frac{\alpha_2 - y\beta_2}{n+\beta_2}\\
&\qquad + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x,y)\left[\frac{y(1-y)}{n+\beta_2} + \frac{(\alpha_2 - y\beta_2)^2}{(n+\beta_2)^2}\right] + (R_{m,n}f)(x,y)\,.
\end{aligned}$$

Then, multiplying the previous inequality by $\min\{m,n\}$ we find

$$\begin{aligned}
&\lim_{m,n\to\infty} \min\{m+\beta_1, n+\beta_2\}\left[S_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f)(x,y) - f(x,y)\right]\\
&\le (\alpha_1 - x\beta_1)\frac{\partial f}{\partial x} + (\alpha_2 - y\beta_2)\frac{\partial f}{\partial y}(x,y)\\
&+\frac{x(1-x)}{2}\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{y(1-y)}{2}\frac{\partial^2 f}{\partial y^2}(x,y)\\
&+\lim_{m,n\to\infty}\min\{m+\beta_1, n+\beta_2\} R_{m,n}. \qquad (2.15)
\end{aligned}$$

Now, we have to prove that

$$\lim_{m,n\to\infty}\min\{m+\beta_1, n+\beta_2\} R_{m,n} = 0.$$

Let $\epsilon > 0$. Since, $\lim\limits_{h\to 0}\mu_1(h) = 0$, there exists an $\delta' > 0$ such that for any $|h| < \delta'$ we have $|\mu_1(h)| < \epsilon$. Since $\lim\limits_{h_1,h_2\to 0}\mu_2(h_1,h_2) = 0$, there exists $\delta'' > 0$ such that for every $|h_1| < \delta''$ and $|h_2| < \delta''$ the inequality $|\mu_2(h_1,h_2)| < \epsilon$ holds. From $\lim\limits_{k\to 0}\mu_3(k) = 0$ we obtain that there exists an $\delta''' > 0$ such that for any k, with $|k| < \delta'''$, we have $\mu_3(k) < \epsilon$. Consider $\delta = \max\{\delta', \delta'', \delta'''\}$. Hence, for every $h, (h_1,h_2), k$ with $|h| < \delta$, $|h_1| < \delta$, $|h_2| < \delta$ and $|k| < \delta$ we get $|\mu_1(h)| < \varepsilon$, $|\mu_2(h_1,h_2)| < \varepsilon$, $|\mu_3(k)| < \varepsilon$.

Let us consider sets

$$\begin{aligned}
I &= \{0,1,2,...,m\}, \\
J &= \{0,1,2,...,n\}, \\
I_1 &= \left\{k \in I; \left|\frac{k+\alpha_1}{m+\beta|_1} - x\right| < \delta\right\}, \\
I_2 &= \left\{k \in I; \left|\frac{k+\alpha_1}{m+\beta_1} - x\right| \geq \delta\right\}, \\
J_1 &= \left\{j \in J; \left|\frac{j+\alpha_2}{n+\beta_2} - y\right| < \delta\right\}, \\
J_2 &= \left\{j \in J; \left|\frac{j+\alpha_2}{n+\beta_2} - y\right| \geq \delta\right\}.
\end{aligned}$$

The maps μ_1, μ_2 and μ_3 are bounded, so that we can write

$$\begin{aligned}
|R_{m,n}| \leq & \sum_{k\in I_1} p_{m,k}(x) \left(\tfrac{k+\alpha_1}{m+\beta_1} - x\right)^2 \left|\mu_1\left(\tfrac{k+\alpha_1}{m+\beta_1} - x\right)\right| \\
& + \sum_{k\in I_2} p_{m,k}(x) \left(\tfrac{k+\alpha_1}{m+\beta_1} - x\right)^2 \left|\mu_1\left(\tfrac{k+\alpha_1}{m+\beta_1} - x\right)\right| \\
& + \sum_{k\in I_1}\sum_{j\in J_1} p_{m,k}(x)\, p_{n,j}(y) \left|\tfrac{k+\alpha_1}{m+\beta_1} - x\right| \cdot \left|\tfrac{j+\alpha_2}{n+\beta_2} - y\right| \mu_2\left(\tfrac{k+\alpha_1}{m+\beta_1} - x, \tfrac{j+\alpha_2}{n+\beta_2} - y\right) \\
& + (\sup|\mu_2|) \left(\sum_{k\in I_1} p_{m,k}(x) \left|\tfrac{k+\alpha_1}{m+\beta_1} - x\right|\right) \left(\sum_{j\in J_2} p_{n,j}(y) \left|\tfrac{j+\alpha_2}{n+\beta_2} - y\right|\right) \\
& + (\sup|\mu_2|) \left(\sum_{k\in I_2} p_{m,k}(x) \left|\tfrac{k+\alpha_1}{m+\beta_1} - x\right|\right) \left(\sum_{j\in J_1} p_{n,j}(y) \left|\tfrac{j+\alpha_2}{n+\beta_2} - y\right|\right) \\
& + (\sup|\mu_2|) \left(\sum_{k\in I_2} p_{m,k}(x) \left|\tfrac{k+\alpha_1}{m+\beta_1} - x\right|\right) \left(\sum_{j\in J_2} p_{n,j}(y) \left|\tfrac{j+\alpha_2}{n+\beta_2} - y\right|\right) \\
& + \sum_{j\in J_1} p_{n,j}(y) \left(\tfrac{j+\alpha_2}{n+\beta_2} - y\right)^2 \left|\mu_3\left(\tfrac{j+\alpha_2}{n+\beta_2} - y\right)\right| \\
& + (\sup|\mu_3|) \sum_{j\in J_2} p_{n,j}(y) \left(\tfrac{j+\alpha_2}{n+\beta_2} - y\right)^2.
\end{aligned} \tag{2.16}$$

For $k \in I_1$ and $j \in J_1$, we have

$$\begin{aligned}
\left|\mu_1\left(\frac{k+\alpha_1}{m+\beta_1} - x\right)\right| &< \varepsilon, \\
\left|\mu_2\left(\frac{k+\alpha_1}{m+\beta_1} - x, \tfrac{j+\alpha_2}{n+\beta_2} - y\right)\right| &< \varepsilon
\end{aligned}$$

and

$$\left|\mu_3\left(\frac{j+\alpha_2}{n+\beta_2} - y\right)\right| < \varepsilon.$$

Moreover we have

$$\sum_{k\in I_1} p_{m,k}(x)\left(\tfrac{k+\alpha_1}{m+\beta_1}-x\right)^2 \le \sum_{k\in I_1} p_{m,k}(x) \le \sum_{k=0}^{m} p_{m,k}(x)=1;$$

$$\left(\sum_{k\in I_1} p_{m,k}(x)\left|\tfrac{k+\alpha_1}{m+\beta_1}-x\right|\right)\left(\sum_{j\in J_2} p_{n,j}(y)\left|\tfrac{j+\alpha_2}{n+\beta_2}-y\right|\right) \le \tfrac{\alpha_1-x\beta_1}{m+\beta_1}\cdot\tfrac{\alpha_2-y\beta_2}{n+\beta_2};$$

$$\left(\sum_{k\in I_2} p_{m,k}(x)\left|\tfrac{k+\alpha_1}{m+\beta_1}-x\right|\right)\left(\sum_{j\in J_1} p_{n,j}(y)\left|\tfrac{j+\alpha_2}{n+\beta_2}-y\right|\right) \le \tfrac{\alpha_1-x\beta_1}{m+\beta_1}\cdot\tfrac{\alpha_2-y\beta_2}{n+\beta_2};$$

$$\left(\sum_{k\in I_2} p_{m,k}(x)\left|\tfrac{k+\alpha_1}{m+\beta_1}-x\right|\right)\left(\sum_{j\in J_2} p_{n,j}(y)\left|\tfrac{j+\alpha_2}{n+\beta_2}-y\right|\right) \le \tfrac{\alpha_1-x\beta_1}{m+\beta_1}\cdot\tfrac{\alpha_2-y\beta_2}{n+\beta_2};$$

$$\sum_{j\in J_1} p_{n,j}(y)\left(\tfrac{j+\alpha_2}{n+\beta_2}-y\right)^2 \le \sum_{j\in J_1} p_{n,j}(y) \le \sum_{j=0}^{m} p_{n,j}(y)=1;$$

$$\sum_{k\in I_1}\sum_{j\in J_1} p_{m,k}(x)\,p_{n,j}(y)\left|\tfrac{k+\alpha_1}{m+\beta_1}-x\right|\cdot\left|\tfrac{j+\alpha_2}{n+\beta_2}-y\right| \le \sum_{k\in I_1}\sum_{j\in J_1} p_{m,k}(x)\,p_{n,j}(y) \le 1.$$

Consequently (2.11) is equivalent to

$$\begin{aligned}|(R_{m,n}f)(x,y)| &\le 3\varepsilon + (\sup|\mu_1|)\sum_{k\in I_2} p_{m,k}(x)\left(\frac{k+\alpha_1}{m+\beta_1}-x\right)^2 \\ &+ 3(\sup|\mu_2|)\frac{\alpha_1-x\beta_1}{m+\beta_1}\cdot\frac{\alpha_2-y\beta_2}{n+\beta_2} + (\sup|\mu_3|)\sum_{j\in J_2} p_{n,j}(y)\left(\frac{j+\alpha_2}{n+\beta_2}-y\right)^2.\end{aligned} \tag{2.17}$$

Now, from the inequality

$$\begin{aligned}\min\{m+\beta_1,n+\beta_2\}&\sum_{k\in I_2} p_{m,k}(x)\left(\frac{k+\alpha_1}{m+\beta_1}-x\right)^2 \\ &\le (m+\beta_1)\sum_{\left|\frac{k+\alpha_1}{m+\beta_1}-x\right|\ge\delta} p_{m,k}(x)\left(\frac{k+\alpha_1}{m+\beta_1}-x\right)^2\end{aligned} \tag{2.18}$$

$$\begin{aligned}\min\{m+\beta_1,n+\beta_2\}&\sum_{j\in J_2} p_{n,j}(y)\left(\frac{j+\alpha_2}{n+\beta_2}-y\right)^2 \\ &\le (n+\beta_2)\sum_{\left|\frac{j+\alpha_2}{n+\beta_2}-y\right|\ge\delta} p_{n,j}(y)\left(\frac{j+\alpha_2}{n+\beta_2}-y\right)^2\end{aligned} \tag{2.19}$$

and Lemma 2.1, we obtain

$$\min_{m,n\to\infty}\{m+\beta_1,n+\beta_2\}\sum_{k\in I_2} p_{m,k}(x)\left(\frac{k+\alpha_1}{m+\beta_1}-x\right)^2=0$$

and

$$\min_{m,n\to\infty}\{m+\beta_1,n+\beta_2\}\sum_{j\in J_2}p_{n,j}(y)\left(\frac{j+\alpha_2}{n+\beta_2}-y\right)^2=0.$$

Hence, there are $m_0', n_0' \in N$ such that for every $m,n\in N$ with $m\geq m_0'$ and $n\geq n_0'$ we have

$$\min\{m+\beta_1,n+\beta_2\}\sum_{k\in I_2}p_{m,k}(x)\left(\frac{k+\alpha_1}{m+\beta_1}-x\right)^2<\epsilon\left(\sup|\mu_1|\right)^{-1} \tag{2.20}$$

and

$$\min\{m+\beta_1,n+\beta_2\}\sum_{j\in J_2}p_{n,j}(y)\left(\frac{j+\alpha_2}{n+\beta_2}-y\right)^2<\varepsilon\left(\sup|\mu_3|\right)^{-1}. \tag{2.21}$$

Since,

$$\lim_{m,n\to\infty}\frac{(\alpha_1-x\beta_1)(\alpha_2-y\beta_2)}{\max\{m+\beta_1,n+\beta_2\}}=0 \tag{2.22}$$

then there exist $m_0'', n_0''\in N$ such that for every $m,n\in N$, with $m\geq m_0''$ and $n\geq n_0''$ we have

$$\frac{(\alpha_1-x\beta_1)(\alpha_2-y\beta_2)}{\max\{m+\beta_1,n+\beta_2\}}<\varepsilon\left(\sup|\mu_2|\right)^{-1}. \tag{2.23}$$

Let $m_0 = \max\{m_0', m_0''\}$ and $n_0 = \max\{n_0', n_0''\}$. Hence for any $m,n\in N$ with $m\geq m_0$ and $n\geq n_0$, by (2.14), (2.15) and (2.17) we can write

$$\min\{m+\beta_1,n+\beta_2\}\left|(R_{m,n}f)(x,y)\right|<\epsilon\left(3\min\{m+\beta_1,n+\beta_2\}+5\right). \tag{2.24}$$

This is equivalent with

$$\min\{m,n\}(R_{m,n}f)(x,y)=0.$$

□

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