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About generalization in mathematics (II)

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ABSTRACT. The goal of this paper is to explore some properties of the set of solutions of a system of equations. We will construct on the set $M_\sigma = \left\{ (x_1, x_2, ..., x_n) \,|x_i \in \mathbb{R},\ x_i = \frac{2x^2_{\sigma(i)}}{1+x^2_{\sigma^{-1}(i)}}, \quad i = \overline{1,n} \right\}$ a boolean ring and we will find a method for determining the subrings of the boolean ring $(\mathcal{P}(X), \Delta, \cap)$, where $\mathcal{P}(X)$ is the collection of all subsets of the set $X = \{1, 2, ..., n\}$.

1. INTRODUCTION

Let S_n denote the symmetric group of a set with n elements and $\sigma \in S_n$. In [1] we have solved the system (S_σ) where:

$$(S_\sigma) : x_i = \frac{2x^2_{\sigma(i)}}{1 + x^2_{\sigma^{-1}(i)}}, \quad i = \overline{1,n} \tag{1}$$

Note

$$M_\sigma = \left\{ (x_1, x_2, ..., x_n) \,|x_i \in \mathbb{R},\ x_i = \frac{2x^2_{\sigma(i)}}{1 + x^2_{\sigma^{-1}(i)}}, \quad i = \overline{1,n} \right\} \tag{2}$$

namely M_σ is the set of solutions of the system (S_σ).

We denote $M = \bigcup\limits_{\sigma \in S_n} M_\sigma$.

Let $X = \{1, 2, ..., n\}$ and $\mathcal{P}(X) = \{Y : Y \subset X\}$ is the set of subsets of X.

In this paper we will determine on the set M_σ a boolean ring and find a method for determining the subrings of the boolean ring $(\mathcal{P}(X), \Delta, \cap)$, where Δ denotes the symmetric difference and $\cap$ denotes the intersection of subsets.

2. MAIN PROPERTIES

Theorem 2.1. *The sets M and $\mathcal{P}(X)$ have the same number of elements, namely 2^n.*

Proof. Let $(x_1, x_2, ..., x_n) \in M$, where $x_i \in \{0, 1\}$. We call the element $(x_1, x_2, ..., x_n)$ a binary word. Let $X = \{1, 2, ..., n\}$ and f the function defined by

$$f : \mathcal{P}(X) \to M, \quad f(X_k) = (x_1, x_2, ..., x_n) \tag{3}$$

where

$$x_i = \begin{cases} 1 & if \quad i \in X_k \\ 0 & if \quad i \notin X_k \end{cases}$$

for all $X_k \subset X$.

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We will prove that f is a bijection. Indeed, for $X_1 \neq X_2$ we have $f(X_1) \neq f(X_2)$ and for all binary words $(x_1, x_2, ..., x_n)$ there exists $X_k \in \mathcal{P}(X)$ such that

$$f(X_k) = (x_1, x_2, ..., x_n).$$

The number of binary words $(x_1, x_2, ..., x_n)$ is 2^n because x_1 can be 0 or 1, x_2 can be 0 or 1 etc. and hence we obtain $2 \cdot 2 \cdot ... \cdot 2 = 2^n$. Also the number of the elements of $\mathcal{P}(X)$ is 2^n. □

Now, we will define on the set M_σ two independent operations, addition and multiplication:

$$(x_1, x_2, ..., x_n) + (x'_1, x'_2, ..., x'_n) = (a_1, a_2, ..., a_n) \tag{4}$$

where for $i = \overline{1, n}$

$$a_i = 0 \text{ if } (x_i = 0 \text{ and } \ x'_i = 0) \text{ or } (x_i = 1 \text{ and } x'_i = 1)$$
$$a_i = 1 \text{ if } (x_i = 1 \text{ and } \ x'_i = 0) \text{ or } (x_i = 0 \text{ and } x'_i = 1)$$

and

$$(x_1, x_2, ..., x_n) \cdot (x'_1, x'_2, ..., x'_n) = (b_1, b_2, ..., b_n) \tag{5}$$

where for $i = \overline{1, n}$

$$b_i = 0 \text{ if } x_i = 0 \text{ or } \ x'_i = 0$$
$$b_i = 1 \text{ if } x_i = 1 \text{ and } x'_i = 1.$$

The set of solution of the system (1) is a stable subset of M with respec to these operations. Using (4) and (5) we easily deduce that the addition and the multiplication are associative and commutative and also related by the distributive law, the addition having the inverse operation of subtraction. Therefore $(M_\sigma, +, \cdot)$ is a ring. Moreover we have

a) $(x_1, x_2, ..., x_n) + (x_1, x_2, ..., x_n) = (0, 0, ..., 0)$;

b) $(x_1, x_2, ..., x_n) \cdot (x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n)$,

hence $(M_\sigma, +, \cdot)$ is a boolean ring with divisors of zero.

c) Because for $\sigma = e$ we obtain $M_e = M$ the algebraic structure $(M, +, \cdot)$ is also a boolean ring.

Hence we have the following result:

Theorem 2.2. *For all $\sigma \in S_n$, the algebraic structure $(M_\sigma, +, \cdot)$ with the operations defined by (4) and (5) is a boolean ring with divisors of zero. In addition, $(M, +, \cdot)$ is also a boolean ring.*

Theorem 2.3. *The rings $(M, +, \cdot)$ and $(\mathcal{P}(X), \Delta, \cap)$ are isomorphic.*

Proof. Let f be the function defined by (3). From Theorem 2.1 we have that f is a bijection. In addition for all $X', X'' \in \mathcal{P}(X)$ we have

$$f\left(X' \Delta X''\right) = f(X') + f(X'') \tag{6}$$

and

$$f\left(X' \cap X''\right) = f(X') \cdot f(X''). \tag{7}$$

Indeed, if $f(X') = (x'_1, x'_2, ..., x'_n)$ and $f(X'') = (x''_1, x''_2, ..., x''_n)$ then,

$$f(X') + f(X'') = (a_1, a_2, ..., a_n)$$

where for $i = \overline{1, n}$, by (4), we have

$$a_i = 0 \text{ if } (x'_i = 0 \text{ and } \; x''_i = 0) \text{ or } (x'_i = 1 \text{ and } x''_i = 1)$$
$$a_i = 1 \text{ if } (x'_i = 1 \text{ and } \; x''_i = 0) \text{ or } (x'_i = 0 \text{ and } x''_i = 1).$$

Therefore, by (3)

$$a_i = 0 \text{ if } (i \in X' \text{ and } i \in X'') \text{ or } (i \notin X' \text{ and } i \notin X'')$$

and

$$a_i = 1 \text{ if } (i \in X' \text{ and } i \notin X'') \text{ or } (i \notin X' \text{ and } i \in X''),$$

so

$$a_i = 0 \text{ if } (i \in X' \cap X'') \text{ or } (i \notin X' \cup X'')$$
$$a_i = 1 \text{ if } (i \in X' \backslash X'') \text{ or } (i \in X'' \backslash X').$$

Hence, by the definition of the symmetric difference we have

$$a_i = \begin{cases} 1 & if \quad i \in X' \Delta X'' \\ 0 & if \quad i \notin X' \Delta X'', \end{cases}$$

such that $(a_1, a_2, ..., a_n) = f(X' \Delta X'')$. Hence

$$f(X') + f(X'') = f\left(X' \Delta X''\right).$$

Similarly we obtain the relation (7). □

Corollary 2.1. *If $\sigma \in S_n$, the symmetric group X, then the restriction of the inverse function of the function f defined by (3) on the subring $(M_\sigma, +, \cdot)$ is carrying in $(\mathcal{P}(X), \Delta, \cap)$ a structure of a subring.*

3. The method for determination of the subrings of the boolean ring $(\mathcal{P}(X), \Delta, \cap)$. An example

By Corollary 2.1. it follows a method for the determination of certain subrings of the boolean ring $(\mathcal{P}(X), \Delta, \cap)$.

Let $\sigma \in S_n$ and M_σ be the set of solutions of system (S_σ) defined by (1). For each element of M_σ, $(x_1, x_2, ..., x_n)$, we determine the subset $X_1 \subset X$, with the property $f(X_1) = (x_1, x_2, ..., x_n)$, namely:

$$\text{if } x_i = 1 \quad \text{then } i \in X_1 \quad \text{and} \quad \text{if } x_i = 0 \quad \text{then } i \notin X_1.$$

Example 3.1. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}.$$

The system (S_σ) is the following

$$(S_\sigma) \begin{cases} x_1 = \dfrac{2x_4^2}{1 + x_2^2} \\ x_2 = \dfrac{2x_1^2}{1 + x_4^2} \\ x_3 = \dfrac{2x_3^2}{1 + x_3^2} \\ x_4 = \dfrac{2x_2^2}{1 + x_1^2} \end{cases}$$

Because σ has two cycles, the number of solutions is $2^2 = 4$. Indeed,

$$M_\sigma = \{I_0, I_1, I_{2,} I_3\}$$

where

$$I_0 = (0,0,0,0),\ I_1 = (1,1,1,1),\ I_2 = (0,0,1,0), I_3 = (1,1,0,1).$$

For addition and multiplication we have the following table:

+	I_0	I_1	I_2	I_3
I_0	I_0	I_1	I_2	I_3
I_1	I_1	I_0	I_3	I_2
I_2	I_2	I_3	I_0	I_1
I_3	I_3	I_2	I_1	I_0

$\cdot$	I_0	I_1	I_2	I_3
I_0	I_0	I_0	I_0	I_0
I_1	I_0	I_1	I_2	I_3
I_2	I_0	I_2	I_2	I_0
I_3	I_0	I_3	I_0	I_3

These operations verify the laws of the ring. In addition we have the following properties:
a) $I_k \cdot I_k = I_k$
b) M_σ have the divisors of zero
c) $I_k + I_k = I_0$.

Now, we find the subset of $X = \{1,2,3,4\}$ carried by $f^{-1} : M \to \mathcal{P}(X)$, where f is the function defined in Theorem 2.1.

$$\begin{aligned} &f^{-1}(I_0) = \oslash, \\ &f^{-1}(I_1) = \{1,2,3,4\}, \\ &f^{-1}(I_2) = \{3\}, \\ &f^{-1}(I_4) = \{1,2,4\}. \end{aligned}$$

It is easy to verify that

$$\mathcal{P}_\sigma(X) = \{\oslash, \{3\}, \{1,2,4\}, \{1,2,3,4\}\}$$

is a subring of $(\mathcal{P}(X), \Delta, \cap)$.

References

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