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On some identities for means in two variables

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ABSTRACT. In this paper we give some identities which are obtained by using the integral representation for some well known means.

1. INTRODUCTION

Let a, b be two positive numbers such that $0 < a < b$. The arithmetic mean of a and b is defined by $A(a,b) = \frac{a+b}{2}$, the geometric mean $G(a,b) = (ab)^{\frac{1}{2}}$, the harmonic mean $H(a,b) = \frac{2ab}{a+b}$, the identric mean $I(a,b) = \frac{1}{e} \cdot (\frac{b^b}{a^a})^{\frac{1}{b-a}}$, and the logarithmic mean $L(a,b) = \frac{b-a}{\ln b - \ln a}$. We also use the power mean of order t for the numbers a and b, that is $A_t(a,b) = \left(\frac{a^t+b^t}{2}\right)^{\frac{1}{t}}$, for $t > 0$. These means are also defined in [3] with some properties. We use the integral representations of some of the means.

$$\ln I(a,b) = \frac{1}{b-a} \int_a^b \ln x dx, \tag{1}$$

$$\frac{1}{L(a,b)} = \frac{1}{b-a} \int_a^b \frac{1}{x} dx, \tag{2}$$

$$A(a,b) = \frac{1}{b-a} \int_a^b x dx, \tag{3}$$

$$\frac{1}{G(a,b)^2} = \frac{1}{b-a} \int_a^b \frac{1}{x^2} dx, \tag{4}$$

$$A_t(a,b) = \left(\frac{t}{b^t - a^t} \int_a^b x^{2t-1} dx\right)^{\frac{1}{t}}, \quad t > 0. \tag{5}$$

Using the property of additivity to the interval for the integral, i.e. $\int_a^b = \int_a^c + \int_c^b$, for $a < c < b$, (and c properly chosen) in $(1) - (5)$, we obtain some identities for means of two variables.

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2. Main results

In this section we obtain some identities that are connecting the means presented above. The first theorem gives a relation between the geometrical and the identric mean of two positive numbers.

Theorem 2.1. *Consider the positive numbers a, b such that $0 < a < b$. With the definitions mentioned above for the identric and for the logarithmic means, the next identity holds:*

$$I(a,b)^{b-a} = I(a, G(a,b))^{G(a,b)-a} \cdot I(G(a,b), b)^{b-G(a,b)}. \tag{6}$$

Proof. From $a < \sqrt{ab} < b$ and (1) we obtain

$$\ln I(a,b) = \frac{1}{b-a}\int_a^b \ln x dx = \frac{1}{b-a}\int_a^{\sqrt{ab}} \ln x dx + \frac{1}{b-a}\int_{\sqrt{ab}}^b \ln x dx,$$

$$\ln I(a,b) = \frac{1}{b-a}\cdot\frac{\sqrt{ab}-a}{\sqrt{ab}-a}\int_a^{\sqrt{ab}} \ln x dx + \frac{1}{b-a}\cdot\frac{b-\sqrt{ab}}{b-\sqrt{ab}}\int_{\sqrt{ab}}^b \ln x dx,$$

so

$$\ln I(a,b) = \frac{\sqrt{ab}-a}{b-a}\cdot \ln I(a,\sqrt{ab}) + \frac{b-\sqrt{ab}}{b-a}\cdot \ln I(\sqrt{ab}, b).$$

This implies

$$(b-a)\ln I(a,b) = (\sqrt{ab}-a)\ln I(a,\sqrt{ab}) + (b-\sqrt{ab})\ln I(b-\sqrt{ab}).$$

Finally this leads to

$$I(a,b)^{b-a} = I(a,\sqrt{ab})^{\sqrt{ab}-a}\cdot I(\sqrt{ab},b)^{b-\sqrt{ab}}$$ □

The following theorem gives a relation between the geometrical, arithmetical and the identric mean of two positive numbers.

Theorem 2.2. *Consider the positive numbers a, b such that $0 < a < b$. Then with the notations mentioned above, we have that the next identity holds:*

$$I(a,b) = G\left(I(a, A(a,b)), I(A(a,b), b)\right). \tag{7}$$

Proof. From $a < \frac{a+b}{2} < b$ and (1) we obtain

$$\frac{1}{b-a}\int_a^b \ln x dx = \frac{\frac{a+b}{2}-a}{b-a}\cdot\frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}} \ln x dx + \frac{b-\frac{a+b}{2}}{b-a}\cdot\frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b \ln x dx,$$

so we have

$$(b-a)\ln I(a,b) = \left(\frac{a+b}{2}-a\right)\ln I\left(a,\frac{a+b}{2}\right) + \left(b-\frac{a+b}{2}\right)\ln I\left(\frac{a+b}{2},b\right),$$

which implies

$$I\left(a,b\right)^{b-a} = I\left(a,\frac{a+b}{2}\right)^{\frac{b-a}{2}}\cdot I\left(\frac{a+b}{2},b\right)^{\frac{b-a}{2}}.$$

Finally this is equivalent to:

$$I(a,b)^2 = I\left(a, \frac{a+b}{2}\right) \cdot I\left(\frac{a+b}{2}, b\right).$$

By applying the square root in the last expression, we get (7) □

From this Theorem, we obtain some nice results for some certain values of a and b.

Proposition 2.1. *If $b = a + 2$ we have that:*

$$I(a, a+2)^2 = I(a, a+1) \cdot I(a+1, a+2) \quad (a > 0)$$

Proposition 2.2. *If $b = a + 1$ we have*

$$I(a, a+1)^2 = I(a, a+\frac{1}{2}) \cdot I(a+\frac{1}{2}, a+1) \quad (a > 0)$$

Proposition 2.3. *If $b = (2n-1)a$ we have that:*

$$I(a, (2n-1)a)^2 = I(a, na) \cdot I(na, (2n-1)a) \quad (a > 0)$$

As a final application we can give the next result.

Proposition 2.4. *Consider that $b = a + 2^n$. Denote by $I_i(a) = I(a+i-1, a+i)$. Then the next identity holds:*

$$I(a, a+2^n)^{2^n} = \prod_{1 \le i \le 2^n} I_i(a)$$

The proof follows easily by the induction and it is let to the reader as an exercise. The following theorem gives a relation between the logarithmic, harmonic and the arithmetical mean of two positive numbers.

Theorem 2.3. *Considering the positive numbers a, b such that $0 < a < b$, the next identity holds:*

$$L(a,b) = H^{-1}\left(L(a, A(a,b)), L(A(a,b), b)\right). \tag{8}$$

We mention that we have considered that

$$H^{-1}(a,b) = \frac{1}{H(a,b)}.$$

Proof. From $a < \frac{a+b}{2} < b$ and (1) we obtain

$$\frac{1}{b-a}\int_a^b \frac{1}{x}dx = \frac{\frac{a+b}{2}-a}{b-a} \cdot \frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}} \frac{1}{x}dx + \frac{b-\frac{a+b}{2}}{b-a} \cdot \frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b \frac{1}{x}dx,$$

This means that

$$\frac{1}{L(a,b)} = \frac{1}{2} \cdot \frac{1}{L(a, \frac{a+b}{2})} + \frac{1}{2} \cdot \frac{1}{L(\frac{a+b}{2}, b)}.$$

From here follows clearly that

$$L(a,b) = H^{-1}\left(L(a, A(a,b)), L(A(a,b), b)\right).$$

□

The following theorem gives a relation between the geometrical, harmonic and the arithmetical mean of two positive numbers.

Theorem 2.4. *Consider the positive numbers a, b such that $0 < a < b$. Then the next identity holds:*

$$G^2(a,b) = H^{-1}\left(G^2(a, A(a,b)), G^2(A(a,b), b)\right). \tag{9}$$

The notations are the ones of the previous theorem.

Proof. From $a < \frac{a+b}{2} < b$ and (4) we obtain

$$\frac{1}{b-a}\int_a^b \frac{1}{x^2}dx = \frac{\frac{a+b}{2}-a}{b-a}\cdot\frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}}\frac{1}{x^2}dx + \frac{b-\frac{a+b}{2}}{b-a}\cdot\frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b \frac{1}{x^2}dx,$$

This means that

$$\frac{1}{G^2(a,b)} = \frac{1}{2}\cdot\frac{1}{G^2(a,\frac{a+b}{2})} + \frac{1}{2}\cdot\frac{1}{G^2(\frac{a+b}{2},b)}.$$

From here follows clearly that

$$G^2(a,b) = H^{-1}\left(G^2(a, A(a,b)), G^2(A(a,b), b)\right).$$

□

The following theorem gives a kind of iteration for the power mean of a, b and power $t > 0$.

Theorem 2.5. *Consider the positive numbers a, b such that $0 < a < b$. Then the next identity holds:*

$$A_t(a,b) = A_t\left(A_t(a, A_t(a,b)), A_t(A_t(a,b), b)\right). \tag{10}$$

Remark 2.1. For $t = 1$ we get $A_1(a,b) = A(a,b)$. For $t = 2$ we get $A_2(a,b) = \left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$ which is named the quadratic mean, another well known mean. It is well known that for fixed positive numbers a, b and real t, the function $t \longmapsto A_t(a,b)$ is increasing, with equality if an only if $a = b$ (see [1] or [2]).

Proof. Because $a < A_t(a,b) < b$ and the relation (5), we obtain:

$$[A_t(a,b)]^t = \frac{t}{b^t - a^t}\int_a^b x^{2t-1}dx$$
$$= \frac{[A_t(a,b)]^t - a^t}{b^t - a^t}\cdot\frac{t}{[A_t(a,b)]^t - a^t}\int_a^{A_t(a,b)} x^{2t-1}dx +$$
$$+\frac{b^t - [A_t(a,b)]^t}{b^t - a^t}\cdot\frac{t}{b^t - [A_t(a,b)]^t}\int_{A_t(a,b)}^b x^{2t-1}dx.$$

But this gives that

$$[A_t(a,b)]^t = \frac{1}{2}\cdot\left[A_t(a, A_t(a,b))^t + A_t(A_t(a,b), b)^t\right].$$

By the definition of $A_t(a,b)$ if follows easily that

$$A_t(a,b) = A_t\left(A_t(a, A_t(a,b)), A_t(A_(a,b), b)\right).$$ □

This ends the proof.

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