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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Some interpolation schemes with triangular and rectangular nodes of Birkhoff type

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ABSTRACT. If P_i, $i = \overline{1,r}$ and Q_j, $j = \overline{1,r}$ are univariate Lagrange interpolation projectors so that the parametric extensions are bivariate projectors which form the chains, i.e.

$$P_1' \leq P_2' \leq ... \leq P_r',\ Q_1'' \leq Q_2'' \leq ... \leq Q_r''$$

the bivariate Biermann interpolation projector is given by

$$B_r = P_1'Q_r'' \oplus ... \oplus P_r'Q_1''$$

(see [8]).

In [3], using chains of bivariate Birkhoff projectors which are parametric extensions of some univariate Birkhoff interpolation projectors, we defined the bivariate Biermann interpolation projector of Birkhoff type. In this article, we give some properties of this projector. The bivariate Biermann interpolation projectors of Birkhoff type with triangular and rectangular nodes are presented. We give representations of these projectors by cardinal functions and we determine the approximation orders. Some numerical examples are given.

1. INTRODUCTION

In this article we construct interpolation schemes using boolean methods and parametric extensions of univariate Birkhoff interpolation projectors. Boolean methods in multivariate approximation were introduced by Gordon W.J. in 1969 in [9]. These results were extended by Delvos F.J., Posdorf H., Schempp W. [6], [7], [8].

Let X, Y be the linear spaces on $\mathbb{R}$.

The linear operator P defined on space X is called projector if $P^2 = P$. The operator $P^C = I - P$, where I is identity operator, is called the remainder projector of P.

The range space of projector P is

$$\mathcal{R}(P) = \{Pf | f \in X\}$$

The set of interpolation points of projector P is denoted by $\mathcal{P}(P)$.

Proposition 1.1. *If P, Q are commutative projectors then we have*

$$\begin{aligned} \mathcal{R}(P \oplus Q) &= \mathcal{R}(P) + \mathcal{R}(Q) \\ \mathcal{P}(P \oplus Q) &= \mathcal{P}(P) \cup \mathcal{P}(Q) \end{aligned} \tag{1.1}$$

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If P_1, P_2 are projectors on space X, we define relation "$\leq$":

$$P_1 \leq P_2 \Leftrightarrow P_1 P_2 = P_1 \tag{1.2}$$

We have the following result from [8].

Proposition 1.2. *Let $r \in \mathbb{N}$, $P_1, \ldots, P_r$ univariate interpolation projectors on $C(X)$ and $Q_1, \ldots, Q_r$ univariate interpolation projectors on $C(Y)$. Let $P_1', \ldots, P_r'$, $Q_1'', \ldots, Q_r''$ be the corresponding parametric extension. We assume that*

$$P_1 \leq P_2 \leq \cdots \leq P_r, \quad Q_1 \leq Q_2 \leq \cdots \leq Q_r \tag{1.3}$$

We have that

$$B_r = P_1' Q_r'' \oplus P_2' Q_{r-1}'' \oplus \cdots \oplus P_r' Q_1'' \tag{1.4}$$

is projector and it has representation

$$B_r = \sum_{m=1}^{r} P_m' Q_{r+1-m}'' - \sum_{m=1}^{r-1} P_m' Q_{r-m}'' \tag{1.5}$$

Moreover, we have

$$B_r^C = P_r'^C + P_{r-1}'^C Q_1''^C + \cdots + P_1'^C Q_{r-1}''^C + Q_r''^C - (P_r'^C Q_1''^C + \cdots + P_1'^C Q_r''^C) \tag{1.6}$$

where $P^C = I - P$, I is identity operator.

Let be $x_k \in [a,b]$, $r_k \in \mathbb{N}$, $N_k \subset \{0, 1, \ldots, r_k\}$, $k = \overline{1,m}$ and $n = |N_1| + |N_2| + \ldots + |N_m| - 1$.

The univariate Birkhoff interpolation operator is an operator $B : X \to \mathbb{P}_n$ which interpolates some derivative (not necessary consecutive) of function $f \in X$ on the set of nodes, i.e.

$$(Bf)^{(j)}(x_k) = f^{(j)}(x_k), \quad k = \overline{1,m}, \quad j \in N_k$$

Remark 1.1. The problem of univariate Birkhoff interpolation has unique solution if corresponding determinant is nonzero.

Particular cases of Birkhoff operator

• a Birkhoff operator $B : X \to \mathbb{P}_n$ which interpolates some consecutive derivative of function $f \in X$ up to some given orders on the set of nodes, i.e.

$$(Bf)^{(j)}(x_k) = f^{(j)}(x_k), \quad k = \overline{1,m}, \quad j = \overline{0, r_k}$$

is called Hermite interpolation operator

• a Birkhoff operator $B : X \to \mathbb{P}_n$ which interpolates the function $f \in X$ on the set of nodes, i.e.

$$(Bf)(x_k) = f(x_k), \quad k = \overline{1,m}$$

is called Lagrange interpolation operator.

In Section 2 we present the Biermann interpolation projector of Birkhoff type from [3]. We prove some properties of this projector in Section 3. In Sections 4 and 5 we develop Biermann operators of Birkhoff type with triangular and respective rectangular elements. These interpolation schemes are useful tools in finite element method.

2. Preliminaries

Let be the univariate Birkhoff interpolation projectors

$$P_1, \ldots, P_r, Q_1, \ldots, Q_r$$

given by

$$\begin{aligned}(P_m f_1)(x) &= \sum_{i=1}^{k_m} \sum_{p \in I_{im}} b_{ip}^m(x) f_1^{(p)}(x_i), \quad 1 \le m \le r \\ (Q_n f_2)(y) &= \sum_{j=1}^{l_n} \sum_{q \in J_{jn}} \widetilde{b}_{jq}^n(y) f_2^{(q)}(y_j), \quad 1 \le n \le r.\end{aligned} \tag{2.7}$$

where $f_1 : [a,b] \to \mathbb{R}$ is a function for which there are $f_1^{(p)}(x_i)$, $i = \overline{1,k_m}$, $p \in I_{im}$, $m = \overline{1,r}$ and $f_2 : [c,d] \to \mathbb{R}$ is a function for which there are $f_2^{(q)}(y_j)$, $j = \overline{1,l_n}$, $q \in J_{jn}$, $n = \overline{1,r}$.

Assume that

$$\{x_1, \ldots, x_{k_m}\} \subseteq [a,b], \quad 1 \le m \le r$$

$$\{y_1, \ldots, y_{l_n}\} \subseteq [c,d], \quad 1 \le n \le r$$

with

$$\begin{array}{l} 1 \le k_1 \le k_2 \le \cdots \le k_r \\ 1 \le l_1 \le l_2 \le \cdots \le l_r \end{array} \tag{2.8}$$

and

$$\begin{array}{ll} I_{im} \subseteq I_{i,m+1}, & i = \overline{1,k_m},\ m = \overline{1,r-1} \\ J_{jn} \subseteq J_{j,n+1}, & j = \overline{1,l_n},\ n = \overline{1,r-1}. \end{array} \tag{2.9}$$

The cardinal functions b_{ip}^m, $m = \overline{1,r}$ and $\widetilde{b}_{jq}^n$, $n = \overline{1,r}$ satisfy the conditions

$$\begin{cases} b_{ip}^{m\ (j)}(x_\nu) = 0,\ \nu \ne i,\ j \in I_{\nu m} \\ b_{ip}^{m\ (j)}(x_i) = \delta_{jp},\ j \in I_{im} \end{cases} \tag{2.10}$$

for $p \in I_{im}$, $\nu, i = \overline{1,k_m}$ and respective

$$\begin{cases} b_{jq}^{n\ (i)}(y_\nu) = 0,\ \nu \ne j,\ i \in J_{\nu n} \\ b_{jq}^{n\ (i)}(y_j) = \delta_{jq},\ i \in J_{jn} \end{cases} \tag{2.11}$$

for $q \in J_{jn}$, $\nu, j = \overline{1,l_n}$.

If $f : [a,b] \times [c,d] \to \mathbb{R}$, than the parametric extensions are given by

$$(P'_m f)(x,y) = \sum_{i=1}^{k_m} \sum_{p \in I_{im}} b_{ip}^m(x) f^{(p,0)}(x_i, y), \quad 1 \le m \le r$$

$$(Q''_n f)(x,y) = \sum_{j=1}^{l_n} \sum_{q \in J_{jn}} \widetilde{b}_{jq}^n(y) f^{(0,q)}(x, y_j), \quad 1 \le n \le r$$

Theorem 2.1. *The parametric extensions*

$$P'_1, \dots, P'_r, Q''_1, \dots, Q''_r$$

are bivariate interpolations projectors which form the chains i.e.

$$P'_1 \le \dots \le P'_r, \quad Q''_1 \le \dots \le Q''_r.$$

The proof of this theorem results using conditions (2.8) and (2.9). The projectors P'_m, Q''_n are commutative

$$P'_m Q''_n = P'_m Q''_n, \quad 1 \le m, n \le r.$$

The tensor product projector $P'_m Q''_n$ has the representation

$$(P'_m Q''_n f)(x,y) = \sum_{i=1}^{k_m} \sum_{p \in I_{im}} \sum_{j=1}^{l_n} \sum_{q \in J_{jn}} b^m_{ip}(x) \widetilde{b}^n_{jq}(y) f^{(p,q)}(x_i, y_j)$$

and interpolation properties

$$(P'_m Q''_n f)^{(p,q)}(x_i, y_j) = f^{(p,q)}(x_i, y_j)$$

$$1 \le i \le k_m, \quad 1 \le j \le l_n, \quad p \in I_{im}, \quad q \in J_{jn}$$

The projectors $P'_1, \dots, P'_r, Q''_1, \dots, Q''_r$ generate a distributive lattice on $C([a,b] \times [c,d])$. A special element of this lattice is

$$B^B_r = P'_1 Q''_r \oplus \dots \oplus P'_r Q''_1, \quad r \in \mathbb{N} \tag{2.12}$$

which is called Biermann interpolation projector of Birkhoff type.

Remark 2.2. If instead of Birkhoff projectors we use Lagrange univariate projectors, i.e.

$$I_{im} = \{0\}, i = \overline{1, k_m}, m = \overline{1, r}$$
$$J_{jn} = \{0\}, j = \overline{1, l_n}, n = \overline{1, r}$$

we obtain Biermann interpolation projector, studied by Delvos F., Posdorf H. [6], [7].

Remark 2.3. The classical Biermann interpolation projector with triangular nodes was given the first by Stancu D.D. in [13]. Triangular grids were used the first by Biermann O.[1] in 1903, but in context cubature.

Remark 2.4. If instead of Birkhoff projectors we use Hermite univariate projectors, i.e.

$$I_{im} = \{0, 1, ..., u_{im}\}, i = \overline{1, k_m}, m = \overline{1, r}$$
$$J_{jn} = \{0, 1, ..., v_{jn}\}, j = \overline{1, l_n}, n = \overline{1, r}$$

we obtain Biermann interpolation projector of Hermite type [2] , [4].

Remark 2.5. In [3] we studied two particular cases of Biermann interpolation projector of Birkhoff type

- Biermann interpolation projector of Abel-Gonciarov type

$$k_m = m, I_{ji} = \{j-1\}, j = \overline{1, r}, i = \overline{j, r}$$
$$l_n = n, J_{ij} = \{i-1\}, i = \overline{1, r}, j = \overline{i, r}$$

• Biermann interpolation projector of Lidstone type

$$k_m = 2, I_{1i} = I_{2i} = \{0, 2, ..., 2i-2\}$$
$$l_n = 2, J_{1j} = J_{2j} = \{0, 2, ..., 2j-2\}$$

3. Properties of Biermann interpolation projector of Birkhoff type

Let $\alpha_i = |I_{1i}| + \cdots + |I_{k_i,i}|$, $\beta_i = |J_{1i}| + \cdots + |J_{l_i,i}|$, $1 \le i \le r$.

Proposition 3.3. *The range space of projector B_r^B is given by*

$$\mathcal{R}(B_r^B) = \Pi_{\alpha_1-1} \otimes \Pi_{\beta_r-1} + \cdots + \Pi_{\alpha_r-1} \otimes \Pi_{\beta_1-1}. \tag{3.13}$$

Proof. Taking into account Proposition 1.1 we have

$$\mathcal{R}(B_r^B) = \mathcal{R}(P_1' Q_r'') + \cdots + \mathcal{R}(P_r' Q_1'').$$

As

$$\mathcal{R}(P_m) = \Pi_{\alpha_m-1}, \quad 1 \le m \le r$$
$$\mathcal{R}(Q_n) = \Pi_{\beta_n-1}, \quad 1 \le n \le r$$

it follows (3.13). □

Proposition 3.4. *The projector B_r^B satisfy interpolation properties*

$$(B_r^B f)^{(p,q)}(x_i, y_j) = f^{(p,q)}(x_i, y_j) \tag{3.14}$$
$$1 \le i \le k_m, \quad 1 \le j \le l_{r+1-m}, \quad 1 \le m \le r$$
$$p \in I_{im} \setminus I_{i,m-1}, \quad q \in J_{r+1-m}$$

where $I_{i,m-1} = \emptyset$, $k_{m-1} < i \le k_m$, $1 \le m \le r$ and $k_0 = 0$.

Proof. From Proposition 1.1 it follows that

$$\mathcal{P}(B_r^B) = \mathcal{P}(P_1' Q_r'') \oplus \cdots \oplus \mathcal{P}(P_r' Q_1'').$$

We denote

$$\mathcal{I}(P) = \{f^{(p,q)}(x_i, y_j) |\ (Pf)^{(p,q)}(x_i, y_j) = f^{(p,q)}(x_i, y_j)\}$$

Then we have

$$\mathcal{I}(B_r) = \mathcal{I}(P_1' Q_r'') \cup \cdots \cup \mathcal{I}(P_r' Q_1'')$$

Next we determine the sets $\mathcal{I}(P_m' Q_{r+1-m}'')$, $1 \le m \le r$.

We have

$$\mathcal{I}(P_1' Q_r'') = \{f^{(p,q)}(x_i, y_j) |\ i = \overline{1, k_1},\ j = \overline{1, l_r},\ p \in I_{i1},\ q \in J_{jr}\}$$
$$= \{f^{(p,q)}(x_i, y_j) |\ i = \overline{1, k_1},\ j = \overline{1, l_r},\ p \in I_{i1} \setminus I_{i0}, \quad q \in J_{jr}\}$$

where $I_{i0} = \emptyset$, $i = \overline{1, k_1}$

For $m = \overline{2, r}$ we have

$$\mathcal{I}(P_m' Q_{r+1-m}'') = \{f^{(p,q)}(x_i, y_j) |\ i = \overline{1, k_m},\ j = \overline{1, l_{r+1-m}},\ p \in I_{im},\ q \in J_{j,r+1-m}\}$$
$$= \{f^{(p,q)}(x_i, y_j) |\ i = \overline{1, k_{m-1}},\ j = \overline{1, l_{r+1-m}},\ p \in I_{i,m-1},\ q \in J_{j,r+1-m}\}$$
$$\cup \{f^{(p,q)}(x_i, y_j) |\ i = \overline{1, k_{m-1}},\ j = \overline{1, l_{r+1-m}},\ p \in I_{im} \setminus I_{i,m-1},\ q \in J_{j,r+1-m}\}$$
$$\cup \{f^{(p,q)}(x_i, y_j) |\ i = \overline{k_{m-1}, k_m},\ j = \overline{1, l_{r+1-m}},\ p \in I_{im},\ q \in J_{j,r+1-m}\}$$
$$= \{f^{(p,q)}(x_i, y_j) |\ i = \overline{1, k_{m-1}},\ j = \overline{1, l_{r+1-m}},\ p \in I_{i,m-1},\ q \in J_{j,r+1-m}\}$$

$$\cup\{f^{(p,q)}(x_i,y_j)|\ i=\overline{1,k_m},\ j=\overline{1,l_{r+1-m}},\ p\in I_{im}\setminus I_{i,m-1},\ q\in J_{j,r+1-m}\}$$
$$\overset{not}{=} A_m\cup C_m$$

where $I_{i,m-1}=\emptyset$, $i=\overline{k_{m-1}+1,k_m}$.

As $A_m\subseteq\mathcal{I}(P'_{m-1}Q_{r+2-m})$, $m=\overline{2,r}$ it follows that

$$\mathcal{I}(B_r)=C_1\cup C_2\cup\cdots\cup C_r,$$

and the sets C_j, $j=\overline{1,r}$ are disjoints. So, relation (3.14) holds. □

Taking into account (1.5) we have

$$(B_r^Bf)(x,y)=\sum_{m=1}^{r}\sum_{i=1}^{k_m}\sum_{p\in I_{im}}\sum_{j=1}^{l_{r+1-m}}\sum_{q\in J_{j,r+1-m}} b_{ip}^m(x)\widetilde{b}_{jq}^{r+1-m}(y)f^{(p,q)}(x_i,y_j) \quad (3.15)$$
$$-\sum_{m=1}^{r-1}\sum_{i=1}^{k_m}\sum_{p\in I_{im}}\sum_{j=1}^{l_{r-m}}\sum_{q\in J_{j,r-m}} b_{ip}^m(x)\widetilde{b}_{jq}^{r-m}(y)f^{(p,q)}(x_i,y_j).$$

From Proposition 3.4 we have the following representation of the projector B_r^B by cardinal functions

$$B_r^Bf=\sum_{m=1}^{r}\sum_{i=1}^{k_m}\sum_{p\in I_{im}\setminus I_{i,m-1}}\sum_{j=1}^{l_{r+1-m}}\sum_{q\in J_{j,r+1-m}}\Phi_{ij}^{pq}f^{(p,q)}(x_i,y_j). \quad (3.16)$$

Proposition 3.5. *The cardinal functions Φ_{ij} are given by formulas*

$$\Phi_{ij}^{pq}(x,y)=\sum_{s\in A_{ij}^{pq}} b_{ip}^s(x)\widetilde{b}_{jq}^{r+1-s}(y)-\sum_{s\in B_{ij}^{pq}} b_{ip}^s(x)\widetilde{b}_{jq}^{r-s}(y) \quad (3.17)$$
$$1\le i\le k_m,\ 1\le j\le l_{r+1-m},\ 1\le m\le r$$
$$p\in I_{im}\setminus I_{i,m-1},\ q\in J_{j,r+1-m}$$

where

$$A_{ij}^{pq}=\{s\in\{1,\dots,r\}|\ i\in X_s,\ p\in I_{is},\ j\in Y_{r+1-s},\ q\in J_{j,r+1-s}\}$$
$$B_{ij}^{pq}=\{s\in\{1,\dots,r-1\}|\ i\in X_s,\ p\in I_{is},\ j\in Y_{r-s},\ q\in J_{j,r-s}\}$$
$$X_s=\{1,\dots,k_s\},\quad Y_s=\{1,\dots,l_s\},\quad 1\le s\le r.$$

Proof. For the function

$$f(x,y)=b_{ip}^r(x)\widetilde{b}_{jq}^r(y).$$

we have

$$B_rf=\Phi_{ij}^{pq}.$$

Taking into account relation (1.5) it follows that

$$\Phi_{ij}^{pq}=\sum_{s=1}^{r}P'_s(b_{ip}^r)\otimes Q''_{r+1-s}(\widetilde{b}_{jq}^r)-\sum_{s=1}^{r-1}P'_s(b_{ip}^r)\otimes Q''_{r-s}(\widetilde{b}_{jq}^r)$$
$$=\sum_{\substack{s\in\{1,\dots,r\}\\ i\in X_s,p\in I_{is}}} b_{ip}^s\otimes Q''_{r+1-s}(\widetilde{b}_{jq}^r)-\sum_{\substack{s\in\{1,\dots,r-1\}\\ i\in X_s,p\in I_{is}}} b_{ip}^s\otimes Q''_{r-s}(\widetilde{b}_{jq}^r)$$

$$= \sum_{\substack{s\in\{1,\dots,r\}\\ i\in X_s, p\in I_{is}\\ j\in Y_{r+1-s}, q\in J_{j,r+1-s}}} b_{ip}^s \otimes \widetilde{b}_{jq}^{r+1-s} - \sum_{\substack{s\in\{1,\dots,r-1\}\\ i\in X_s, p\in I_{is}\\ j\in Y_{r-s}, q\in J_{j,r-s}}} b_{ip}^s \otimes \widetilde{b}_{jq}^{r-s}.$$

□

Proposition 3.6. *If $f \in C^{\alpha_r,\beta_r}([a,b]\times[c,d])$ we can give the following representation for the remainder term in Biermann interpolation of Birkhoff type.*

$$f(x,y) - (B_r^B f)(x,y) \tag{3.18}$$

$$= \int_a^b \varphi_{\alpha_r}(x,s) f^{(\alpha_r,0)}(s,y)ds + \int_c^d \psi_{\beta_r}(y,t) f^{(0,\beta_r)}(x,t)dt$$

$$+ \sum_{m=1}^{r-1} \int_a^b \int_c^d \varphi_{\alpha_{r-m}}(x,s)\psi_{\beta_m}(y,t) f^{(\alpha_{r-m},\beta_m)}(s,t)dsdt$$

$$- \sum_{m=1}^{r} \int_a^b \int_c^d \varphi_{\alpha_{r+1-m}}(x,s)\psi_{\beta_m}(y,t) f^{(\alpha_{r+1-m},\beta_m)}(s,t)dsdt$$

where the kernel functions are given by relations

$$\varphi_{\alpha_m}(x,s) = P_m^c\left[\frac{(x-s)_+^{\alpha_m-1}}{\alpha_m!}\right], \quad 1 \le m \le r$$

$$\psi_{\beta_n}(y,t) = Q_n^c\left[\frac{(y-t)_+^{\beta_n-1}}{\beta_n!}\right], \quad 1 \le n \le r$$

($z_+ = 0$, if $z < 0$ and $z_+ = z$ if $z \ge 0$).

Proof. If $f_1 \in C^{\alpha_m}[a,b]$ şi $f_2 \in C^{\beta_n}[c,d]$ then

$$(P_m^c f_1)(x) = f_1(x) - (P_m f_1)(x) = \int_a^b \varphi_{\alpha_m}(x,s) f_1^{(\alpha_m)}(s)ds, \quad 1 \le m \le r$$

$$(Q_n^c f_2)(y) = f_2(y) - (Q_n f_2)(y) = \int_c^d \varphi_{\beta_n}(y,t) f_2^{(\beta_n)}(t)dt, \quad 1 \le n \le r.$$

Using formula (1.6) we get (3.18). □

Proposition 3.7. *We assume that $|a|,|b|,|c|,|d| \le 1$ and $I_k \subset \{0,1\}$. Let $h = b-a = d-c$ and $q = \min\{\alpha_{r-m}+\beta_m,\ 0 \le m \le r\}$ with $\alpha_0 = 0$, $\beta_0 = 0$. Than we have*

$$f(x,y) - (B_r^B f)(x,y) = O(h^q), \quad h \to 0. \tag{3.19}$$

Proof. Taking into account

$$f_1(x) - (P_m f_1)(x) = O(h^{\alpha_m}), \quad 1 \le m \le r$$

$$f_2(y) - (Q_n f_2)(y) = O(h^{\beta_n}), \quad 1 \le n \le r$$

and using formula (1.6) we get (3.19). □

4. Triangular elements

Let be the interpolation nodes

$$x_i = \frac{(i-1)h}{(r-1)}, \quad y_j = \frac{(j-1)h}{(r-1)}, \quad 1 \le i, j \le r, \quad h > 0$$

and

$$k_m = m,\ l_n = n,\ 1 \le m, n \le r$$
$$I_{1m} = \{0\},\ 1 \le m \le r$$
$$I_{im} = \{1\},\ 2 \le i \le m,\ 2 \le m \le r$$
$$J_{1n} = \{0\},\ 1 \le n \le r$$
$$J_{jn} = \{1\},\ 2 \le j \le n,\ 2 \le n \le r.$$

Let be the univariate Birkhoff interpolation projectors

$$(P_m f_1)(x) = \sum_{i=1}^{m} \sum_{p \in I_{im}} b_{ip}^m(x) f_1^{(p)}(x_i), \quad 1 \le m \le r$$

$$(Q_n f_2)(y) = \sum_{j=1}^{n} \sum_{q \in J_{jn}} \widetilde{b}_{jq}^n(y) f_2^{(q)}(y_j), \quad 1 \le n \le r$$

where the functions b_{ip}^m and $\widetilde{b}_{jq}^n$ are obtained from conditions (2.10) and respective (2.11).

As the conditions (2.8) and (2.9) are satisfied, it follows that the parametric extensions are projectors which form the chains

$$P_1' \le \ldots \le P_r', \quad Q_1'' \le \ldots \le Q_r''$$

The Biermann interpolation operator of Birkhoff type is defined by

$$B_r^B = P_1' Q_r'' \oplus \ldots \oplus P_r' Q_1''.$$

The projector B_r^B has the following interpolation properties

$$(B_r^B f)(x_1, y_1) = f(x_1, y_1),$$
$$(B_r^B f)^{(1,0)}(x_i, y_1) = f^{(1,0)}(x_i, y_1), \quad 2 \le i \le r,$$
$$(B_r^B f)^{(0,1)}(x_1, y_j) = f^{(0,1)}(x_1, y_j), \quad 2 \le j \le r,$$
$$(B_r^B f)^{(1,1)}(x_i, y_j) = f^{(1,1)}(x_i, y_j), \quad 2 \le i \le r, \quad 2 \le j \le r+1-i.$$

The range space of the projector B_r^B is

$$\mathcal{R}(B_r^B) = \Pi_0 \otimes \Pi_{r-1} + \Pi_1 \otimes \Pi_{r-2} + \ldots + \Pi_{r-1} \otimes \Pi_0.$$

We have the following representation of projector B_r^B by cardinal functions

$$B_r^B f = \sum_{m=1}^{r} \sum_{j=1}^{r+1-m} \sum_{p \in I_{m,m}} \sum_{q \in J_{j,r+1-m}} \Phi_{mj}^{pq} f^{(p,q)}(x_m, y_j) \tag{4.20}$$

where

$$\Phi_{mj}^{pq}(x, y) = \sum_{s=m}^{r+1-j} b_{mp}^s(x) \widetilde{b}_{jq}^{r+1-s}(y) - \sum_{s=m}^{r-j} b_{mp}^s(x) \widetilde{b}_{jq}^{r-s}(y)$$

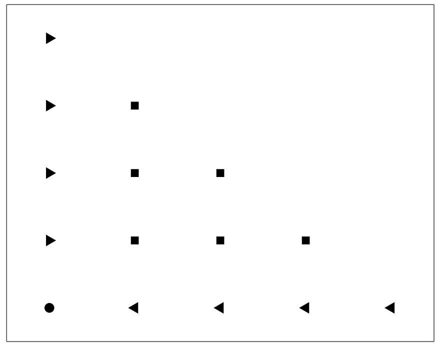

FIGURE 1. Triangular elements $r = 5$

$$1 \le m \le r, \quad 1 \le j \le r+1-m, \quad p \in I_{mm}, \quad q \in J_{j,r+1-m}.$$

The approximation order is r, i.e.

$$f(x,y) - (B_r^B f)(x,y) = O(h^r), \quad h \to 0. \tag{4.21}$$

We approximate the function $f : T_1 \to \mathbb{R}$ $f(x,y) = 1/(1+x^2+y^2)$ using Biermann interpolation of Birkhoff type. In tabel are given the estimations of error in max-norm.

r	$\|\|f - B_r^B f\|\|$
2	0.16666666666667
3	0.24232025881303
4	0.03580421497499
5	0.02297335750323

5. RECTANGULAR ELEMENTS

Let be the interpolation nodes

$$x_{2i-1} = -\frac{(1+2(r-i))h}{2(2r-1)}, \quad x_{2i} = -x_{2i-1}, \quad 1 \le i \le r$$

$$x_{2j-1} = -\frac{(1+2(r-j))h}{2(2r-1)}, \quad x_{2j} = -x_{2j-1}, \quad 1 \le j \le r$$

and

$$k_m = 2m,\ l_n = 2n,\ 1 \le m,n \le r$$

$$I_{1m} = \{0\},\ 1 \le m \le r$$

$$I_{im} = \{1\},\ 2 \le i \le 2m,\ 1 \le m \le r$$

$$J_{1n} = \{0\},\ 1 \le n \le r$$

$$J_{jn} = \{1\},\ 2 \le j \le 2n,\ 1 \le n \le r.$$

The univariate interpolation Birkhoff projectors are given by

$$(P_m f_1)(x) = \sum_{i=1}^{2m} \sum_{p\in I_{im}} b_{ip}^m(x) f_1^{(p)}(x_i), \quad 1 \le m \le r$$

$$(Q_n f_2)(y) = \sum_{j=1}^{2n} \sum_{q\in J_{jn}} \widetilde{b}_{jq}^n(y) f_2^{(q)}(y_j),\ 1 \le n \le r$$

where the cardinal functions b_{ip}^m and $\widetilde{b}_{jq}^n$ are obtained by conditions (2.10) and respective (2.11).

As the conditions (2.8) and (2.9) are satisfied, it follows that the parametric extensions are bivariate projectors which form the chains

$$P_1' \le ... \le P_r', \quad Q_1'' \le ... \le Q_r''$$

The Biermann interpolation projector of Birkhoff type is given by

$$B_r^B = P_1' Q_r'' \oplus ... \oplus P_r' Q_1''.$$

The projector B_r^B has the interpolation properties

$$(B_r^B f)(x_1, y_1) = f(x_1, y_1),$$

$$(B_r^B f)^{(1,0)}(x_i, y_1) = f^{(1,0)}(x_i, y_1), \quad 2 \le i \le 2r,$$

$$(B_r^B f)^{(0,1)}(x_1, y_j) = f^{(0,1)}(x_1, y_j), \quad 2 \le j \le 2r,$$

$$(B_r^B f)^{(1,1)}(x_i, y_j) = f^{(1,1)}(x_i, y_j),$$

$$2 \le i \le 2m, \quad 2 \le j \le 2(r+1-m), \quad 1 \le m \le r.$$

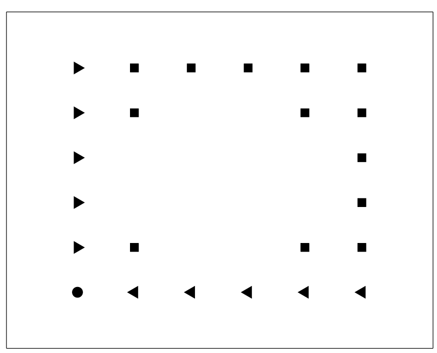

FIGURE 2. Rectangular elements $r = 3$

The range space of projector B_r^B is

$$\mathcal{R}(B_r^B) = \Pi_1 \otimes \Pi_{2r-1} + \Pi_3 \otimes \Pi_{2r-3} + ... + \Pi_{2r-1} \otimes \Pi_1.$$

We have the following representations of the projector B_r^B by cardinal functions

$$B_r^B f = \sum_{m=1}^{r} \sum_{i=2m-1}^{2m} \sum_{j=1}^{2(r+1-m)} \sum_{p\in I_{im}} \sum_{q\in J_{j,r+1-m}} \Phi_{ij}^{pq} f^{(p,q)}(x_i, y_j) \tag{5.22}$$

where

$$\Phi_{ij}^{pq}(x,y) = \sum_{s=[\frac{i-1}{2}]+1}^{[r+1-\frac{j}{2}]} b_{ip}^s(x)\widetilde{b}_{jq}^{r+1-s}(y) - \sum_{s=[\frac{i-1}{2}]+1}^{[r-\frac{j}{2}]} b_{ip}^s(x)\widetilde{b}_{jq}^{r-s}(y)$$

$$1 \le i \le 2m, \quad 1 \le j \le 2(r+1-m), 1 \le m \le r, \quad p \in I_{im},\ q \in J_{j,r+1-m}.$$

The approximation order is $2r$, i.e.

$$f(x,y) - (B_r^B f)(x,y) = O(h^{2r}), \quad h \to 0. \tag{5.23}$$

We approximate the function $f : [-1/2, 1/2] \times [-1/2, 1/2] \to \mathbb{R}$, $f(x,y) = 1/(1+x^2+y^2)$ using Biermann interpolation of Birkhoff type. In table are given the estimations of error in max-norm.

r	$\left\|f - B_r^B f\right\|$
2	0.69160966670624
3	0.04507288326815
4	0.02652021446710

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