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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Modified Jakimovski-Leviatan operators

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ABSTRACT. In this paper we modify the Jakimovski-Leviatan operators, in order to improve the rate of convergence. We estimate the order of approximation and we give a Voronovskaya type theorem.

1. INTRODUCTION

In 1969, A. Jakimovski and D. Leviatan [4] introduced a Favard-Szasz type operator, as follows: if $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(1) \neq 0$ is an analytic function in the disk $|z| < R$, $R > 1$, let p_k be the Appell polynomials defined by the relation

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.1}$$

Therefore, the polynomials are

$$p_k(x) = \sum_{\nu=0}^{k} a_\nu \frac{x^{k-\nu}}{(k-\nu)!}, \quad k \in \mathbb{N}.$$

Let E be the class of functions of exponential type, which satisfy the property $|f(t)| \leq ce^{At}$, $(t \geq 0)$ for some finite constants $c, A > 0$. In [4], the authors considered the operator $P_n : E \to C[0, \infty)$

$$P_n(f; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}. \tag{1.2}$$

Remark. If $g(z) \equiv 1$, by (1) we obtain $p_k(x) = \frac{x^k}{k!}$ and we obtain Szasz-Mirakjan operators:

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

A. Jakimovski and D. Leviatan established the analogue to Szasz's results, as well as certain other approximation theorems. B. Wood [6] proved that the operators P_n are positive if and only if

$$\frac{a_k}{g(1)} \geq 0, (k = 0, 1, \dots).$$

Throughout this paper we will assume that the operators P_n are positive.

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In [2] was studied the rate of convergence of the sequence $(P_n f)$ to f:
If $f \in E$ and $f \in C[0,a]$, then

$$|P_n(f;x) - f(x)| \leq \left(1 + \sqrt{a + \frac{1}{n} \cdot \frac{g''(1)+g'(1)}{g(1)}}\right) \omega\left(f; \frac{1}{\sqrt{n}}\right),$$

where ω is the first order modulus of continuity of f.

Ispir Nurhayat [3] investigated the approximation of continuous functions having polynomial growth at infinity, by the operator given in (1.2). Ulrich Abel and Mircea Ivan [1], gave an asymptotic expansion of the operators P_n and their derivatives.

2. The modified operator

In this paper we will modify the operator (1.2), in order to improve the rate of convergence of the sequence $(P_n f)$ to f.

Let C_B be the set of all real-valued functions f uniformly continuous and bounded on $[0,\infty)$ with the norm defined by

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|. \tag{2.3}$$

For a fixed $r \in \mathbb{N} = \{0, 1, \dots\}$ we denote by

$$C_B^r = \{f \in C_B \text{ such as } f', \dots, f^{(r)} \in C_B\}.$$

The norm in C_B^r is given also by (2.3).

Let $r \in \mathbb{N}$ be a fixed number. For $f \in C_B^r$, $x \in [0,\infty)$ and $n \in \mathbb{N}$ we define the operators

$$P_{n,r}(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r} \frac{f^{(j)}\left(\dfrac{k}{n}\right)}{j!} \left(x - \frac{k}{n}\right)^j, \tag{2.4}$$

where p_k are Appell polynomials defined by (1.1).

Remarks. 1. For $r = 0$, we have $P_{n,0}(f;x) = P_n(f;x)$.

2. $P_{n,0}(e_0;x) = P_n(e_0;x) = 1$, where $e_0(x) = 1$, $x \geq 0$.

Lemma 2.1.
For every fixed $q \leq r$, $q \in \mathbb{N}$, we have

$$P_{n,r}(e_q;x) = x^q, \text{ where } e_q(x) = x^q, \quad x \in [0,\infty).$$

Proof. For $q \leq r$ and $x \in [0, \infty)$, we have

$$P_{n,r}(e_q; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{q} \frac{q(q-1)\dots(q-j+1)}{j!} \left(\frac{k}{n}\right)^{q-j} \left(x - \frac{k}{n}\right)^j$$

$$= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{q} \binom{q}{j} \left(\frac{k}{n}\right)^{q-j} \left(x - \frac{k}{n}\right)^j$$

$$= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) x^q = x^q P_n(e_0; x) = x^q.$$

In order to prove the main results we will use a result due to U. Abel and M. Ivan [1], concerning Jakimovski-Leviatan operators:

Lemma A. *For each $x \geq 0$ and all $s = 0, 1, 2, \dots$, the central moments of the operators P_n satisfy the estimation*

$$P_n((\cdot - x)^s; x) = \mathcal{O}\left(n^{-\left[\frac{s+1}{2}\right]}\right), \quad (n \to \infty),$$

where $[s]$ denotes the integral part of s.

3. Main results

We will estimate the order of approximation of a function $f \in C_B^r$ by the sequence $(P_{n,r} f)$, using the first order modulus of continuity:

$$\omega(f; \delta) = \sup_{\substack{0 \leq h \leq \delta \\ x \in [0,\infty)}} \| f(x+h) - f(x) \|.$$

Theorem 3.1. *Let $r \in \mathbb{N}$ be a fixed number. There exists a positive constant $\mathcal{C}(r)$, depending only on r, such that*

$$|P_{n,r}(f; x) - f(x)| \leq \mathcal{C}(r) \frac{1}{\sqrt{n^{\left[r+\frac{1}{2}\right]}}} \omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right),$$

for every $f \in C_B^r$ and $n \in \mathbb{N} \setminus \{0\}$.

Proof. For $r = 0$, the inequality is known. Let $f \in C_B^r$, $r \geq 1$ and $y \in [0, \infty)$ be a fixed point. We apply the following modified Taylor formula

$$f(x) = \sum_{j=0}^{r} \frac{f^{(j)}(y)}{j!} (x-y)^j$$

$$+ \frac{(x-y)^r}{(r-1)!} \int_0^1 (1-t)^{r-1} \{ f^{(r)}(y + t(x-y)) - f^{(r)}(y) \} dt, \quad x \in [0, \infty).$$

We choose $y = \dfrac{k}{n}$, for fixed $k \in \mathbb{N}$ and $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and we have

$$f(x) = P_{n,r}(1; x) f(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f(x)$$

$$= P_{n,r}(f;x) + \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{\left(x - \frac{k}{n}\right)^r}{(r-1)!}$$

$$\times \int_0^1 (1-t)^{r-1} \left\{ f^{(r)}\left(\frac{k}{n} + t\left(x - \frac{k}{n}\right)\right) - f^{(r)}\left(\frac{k}{n}\right) \right\} dt.$$

By the properties of the modulus of continuity, for $t \in [0,1]$, we have

$$\left| f^{(r)}\left(\frac{k}{n} + t\left(x - \frac{k}{n}\right)\right) - f^{(r)}\left(\frac{k}{n}\right) \right| \le \omega\left(f^{(r)}; t\left|x - \frac{k}{n}\right|\right)$$

$$\le \omega\left(f^{(r)}; \left|x - \frac{k}{n}\right|\right) = \omega\left(f^{(r)}; \sqrt{n}\left|x - \frac{k}{n}\right| \frac{1}{\sqrt{n}}\right)$$

$$\le \left(1 + \sqrt{n}\left|x - \frac{k}{n}\right|\right) \omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right).$$

It results that

$$|f(x) - P_{n,r}(f;x)| \le \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{\left|x - \frac{k}{n}\right|^r}{(r-1)!}$$

$$\times \int_0^1 (1-t)^{r-1} \left(1 + \sqrt{n}\left|x - \frac{k}{n}\right|\right) \omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right) dt$$

$$= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{\left|x - \frac{k}{n}\right|^r}{r!} \left(1 + \sqrt{n}\left|x - \frac{k}{n}\right|\right) \omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right).$$

Therefore,

$$|f(x) - P_{n,r}(f;x)| \le \frac{1}{r!}\omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right) \{P_n(|x-t|^r; x) + \sqrt{n} P_n(|x-t|^{r+1}; x)\}.$$

Next we use Cauchy's inequality and Lemma A, and we obtain

$$P_n(|x-t|^r; x) \le \{P_n((x-t)^{2r}; x)\}^{1/2} \le \left\{\frac{M_1(r)}{n^{[(2r+1)/2]}}\right\}^{1/2}$$

and

$$P_n(|x-t|^{r+1}; x) \le \{P_n((x-t)^{2r+2}; x)\}^{1/2} \le \left\{\frac{M_2(r)}{n^{[(2r+3)/2]}}\right\}^{1/2},$$

where $M_1(r)$, $M_2(r)$ are positive constants, depending on r. It results that

$$|f(x) - P_{n,r}(f;x)| \le \frac{1}{r!}\omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right) \left\{ \left(\frac{M_1(r)}{n^{\left[r+\frac{1}{2}\right]}}\right)^{1/2} + \sqrt{n}\left(\frac{M_2(r)}{n^{\left[r+1+\frac{1}{2}\right]}}\right)^{1/2} \right\}$$

$$= \frac{1}{r!}\omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right)\left\{\left(\frac{M_1(r)}{n^{\left[r+\frac{1}{2}\right]}}\right)^{1/2} + \left(\frac{M_2(r)}{n^{\left[r+\frac{1}{2}\right]}}\right)^{1/2}\right\}$$

$$= \mathcal{C}(r)\frac{1}{\sqrt{n^{\left[r+\frac{1}{2}\right]}}}\omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right).$$

Corollary. *If* $f \in C_B^r$, $r \in \mathbb{N}^*$, *then*

$$\lim_{n\to\infty} |P_{n,r}(f;x) - f(x)| = 0, \quad x \in [0,\infty).$$

Theorem 3.2. *If* $f \in C_B^{r+2}$, *for a fixed* $r \in \mathbb{N}$, *then for every* $x \in [0,\infty)$ *we have*

$$P_{n,r}(f;x) - f(x) = (-1)^r \frac{f^{(r+1)}(x)}{(r+1)!}P_n((t-x)^{r+1};x)$$

$$+(-1)^r(r+1)\frac{f^{(r+2)}(x)}{(r+2)!}P_n((t-x)^{r+2};x) + o\left(\frac{1}{n\sqrt{n^{\left[r+\frac{1}{2}\right]}}}\right), \quad \textit{as} \quad n \to \infty.$$

*P*roof. If $x = 0$, we have $P_{n,r}(f;0) = f(0)$, $n \in \mathbb{N}^*$, $r \in \mathbb{N}$.

Let $x > 0$ be. The function $f^{(j)} \in C_B^{r+2-j}$, $0 \le j \le r$, so we can apply Taylor formula

$$f^{(j)}(x) = \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!}(t-x)^i + \varphi_j(t;x)(t-x)^{r+2-j},$$

where $t \in [0,\infty)$ and $\varphi_j(t;x) \equiv \varphi_j(t)$ is a function such that $\varphi_j(t)t^{r+2-j} \in C_B^{r+2-j}$ and $\lim_{t\to x} \varphi_j(t) = 0$. We choose $t = \dfrac{k}{n}$, we replace $f^{(j)}$ in (2.4) and we obtain

$$P_{n,r}(f;x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\sum_{j=0}^{r}\frac{\left(x-\frac{k}{n}\right)^j}{j!}\sum_{i=0}^{r+2-j}\frac{f^{(j+i)}(x)}{i!}\left(\frac{k}{n}-x\right)^i$$

$$+\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\sum_{j=0}^{r}\frac{\left(x-\frac{k}{n}\right)^j}{j!}\varphi_j\left(\frac{k}{n};x\right)\left(\frac{k}{n}-x\right)^{r+2-j}$$

$$= A_{n,r}(x) + B_{n,r}(x).$$

We have

$$A_{n,r}(x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\sum_{j=0}^{r}\frac{\left(x-\frac{k}{n}\right)^j}{j!}\sum_{l=j}^{r+2}\frac{f^{(l)}(x)}{(l-j)!}\left(\frac{k}{n}-x\right)^{l-j}$$

$$= \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\sum_{j=0}^{r}\frac{(-1)^j}{j!}\left\{\sum_{l=j}^{r}\frac{f^{(l)}(x)}{(l-j)!}\left(\frac{k}{n}-x\right)^l\right.$$

$$\left.+\frac{f^{(r+1)}(x)}{(r+1-j)!}\left(\frac{k}{n}-x\right)^{r+1} + \frac{f^{(r+2)}(x)}{(r+2-j)!}\left(\frac{k}{n}-x\right)^{r+2}\right\} = S_1 + S_2 + S_3.$$

We observe that

$$S_1 = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \sum_{l=0}^{r} \frac{f^{(l)}(x)}{l!} \left(\frac{k}{n} - x\right)^l \sum_{j=0}^{l} \binom{l}{j} (-1)^j = f(x),$$

because $\sum_{j=0}^{l} (-1)^j \binom{l}{j} = 0$ for $l > 0$

$$S_2 = \frac{f^{(r+1)}(x)}{(r+1)!} \cdot \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \left(\frac{k}{n} - x\right)^{r+1} \sum_{j=0}^{r} (-1)^j \binom{r+1}{j}.$$

But $\sum_{j=0}^{r} (-1)^j \binom{r+1}{j} = (-1)^r$, therefore

$$S_2 = (-1)^r \frac{f^{(r+1)}(x)}{(r+1)!} (P_n(t-x)^{r+1}; x).$$

In the same way, we obtain

$$S_3 = (-1)^r (r+1) \frac{f^{(r+2)}(x)}{(r+2)!} P_n((t-x)^{r+2}; x).$$

It results that

$$A_{n,r}(x) = f(x) + (-1)^r \frac{f^{(r+1)}(x)}{(r+1)!} P_n((t-x)^{r+1}; x) \qquad (3.5)$$
$$+ (-1)^r (r+1) \frac{f^{(r+2)}(x)}{(r+2)!} P_n((t-x)^{r+2}; x).$$

For $B_{n,r}$, we have

$$B_{n,n}(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \left(\frac{k}{n} - x\right)^{r+2} \sum_{j=0}^{r} (-1)^j \frac{1}{j!} \varphi_j \left(\frac{k}{n}; x\right).$$

We denote by

$$\phi_r(t) = \sum_{j=0}^{r} (-1)^j \frac{1}{j!} \varphi_j(t; x), \quad t \in [0, \infty)$$

and we can write

$$B_{n,r}(x) = P_n((t-x)^{r+2} \phi_r(t); x).$$

By Cauchy's inequality and Lemma A, we obtain

$$|B_{n,r}(x)| \le \{P_n((t-x)^{2r+4}; x)\}^{1/2} \{P_n(\phi_r^2(t); x)\}^{1/2}$$

$$\le \left\{ M_1(r) \frac{1}{n^{[(2r+5)/2]}} \right\}^{1/2} \{P_n(\phi_r^2(t); x)\}^{1/2}$$

$$= M(r) \frac{1}{n\sqrt{n^{[r+\frac{1}{2}]}}} \{P_n(\phi_r^2(t);x)\}^{1/2},$$

where $M_1(r)$, $M(r)$ are positive constants depending on r.

But

$$\lim_{n\to\infty} P_n(\phi_r^2(t);x) = \phi_r^2(x) = 0$$

therefore

$$B_{n,r}(x) = o\left(\frac{1}{n\sqrt{n^{[r+\frac{1}{2}]}}}\right), \quad n \to \infty. \tag{3.6}$$

By (3.5), (3.6) we get to the desired result.

Remark. Let us consider $r = 0$ in Theorem 3.2. We obtain: if $f \in C_B^2$, then

$$P_n(f;x) - f(x) = f'(x)P_n((t-x);x) + \frac{f''(x)}{2} P_n((t-x)^2;x) + o\left(\frac{1}{n}\right).$$

We replace here

$$P_n(t-x;x) = \frac{1}{n} \cdot \frac{g'(1)}{g(1)}$$

$$P_n((t-x)^2;x) = \frac{x}{n} + \frac{1}{n^2} \cdot \frac{g''(1) + g'(1)}{g(1)},$$

and we obtain the Voronovskaya Theorem for the Jakimovski-Leviatan operators:

$$\lim_{n\to\infty} n[P_n(f;x) - f(x)] = \frac{g'(1)}{g(1)} f'(x) + \frac{x}{2} f''(x).$$

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