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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Differentiability with respect to delays for a Lotka-Volterra system*

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ABSTRACT. We study the differentiability with respect to delays using the weakly Picard operators technique.

1. INTRODUCTION

Consider the following Lotka-Volterra differential system with delays

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)),\ i = 1,2,\ t \in [t_0, b] \tag{1.1}$$

$$\begin{cases} x_1(t) = \varphi(t),\ t \in [t_0 - \tau_1, t_0], \\ x_2(t) = \psi(t),\ t \in [t_0 - \tau_2, t_0]. \end{cases} \tag{1.2}$$

Suppose that we have satisfied the following conditions:

(H_1) $t_0 < b,\ \tau, \tau_1, \tau_2 > 0,\ \tau_1 < \tau_2 < \tau,\ \tau_1, \tau_2 \in J,\ J = [t_0, \tau]$ a compact interval;

(H_2) $f_i \in C^1([t_0, b] \times \mathbb{R}^4, \mathbb{R}),\ i = 1,2$;

(H_3) there exists $L_f > 0$ such that

$$\left\| \frac{\partial f_i}{\partial u_j}(t, u_1, u_2, u_3, u_4) \right\|_{\mathbb{R}} \leq L_f,$$

for all $t \in [t_0, b],\ u_j \in R,\ j = \overline{1,4},\ i = 1,2$;

(H_4) $\varphi \in C([t_0 - \tau, t_0], \mathbb{R}),\ \psi \in C([t_0 - \tau, t_0], \mathbb{R})$;

In the above conditions, from the Theorem 1, in [4], we have that the problem (1.1)–(1.2) has a unique solution, $(x_1(t), x_2(t))$.

2. WEAKLY PICARD OPERATORS

In this paper we need some notions and results from the weakly Picard operator theory (for more details see I. A. Rus [9], [8], M. Şerban [14]).

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ - the family of the nonempty invariant subset of A;

$A^{n+1} := A \circ A^n,\ A^0 = 1_X,\ A^1 = A,\ n \in \mathbb{N}$ - the iterant operators of A, where 1_X is the identity operator;

$P(X) := \{Y \subset X \mid Y \neq \emptyset\}$ - the set of the parts of X.

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Definition 2.1. Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:
(i) $F_A = \{x^*\}$;
(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.2. Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.

Theorem 2.1. *Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda$$

such that:
(a) $X_\lambda \in I(A),\ \lambda \in \Lambda,\ I(A)$-the family of nonempty invariant subsets of A;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard (c-Picard) operator for all $\lambda \in \Lambda$.

Theorem 2.2. (Fibre contraction principle). *Let (X, d) and (Y, ρ) be two metric spaces and $A : X \times X \to X \times X,\ A = (B, C),\ (B : X \to X,\ C : X \times Y \to Y)$ a triangular operator. We suppose that*
(i) (Y, ρ) is a complete metric space;
(ii) the operator B is PO;
(iii) there exists $L \in [0, 1)$ such that $C(x, \cdot) : Y \to Y$ is a L-contraction, for all $x \in X$;
(iv) if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.
Then the operator A is PO.

3. Main result

Now we prove that

$$x_i(t, \cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau, b],\ i = 1, 2.$$

For this we consider the system

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(t - \tau_1), x_2(t - \tau_2)),\ i = 1, 2 \tag{3.3}$$

where $t \in [t_0, b],\ x_1 \in C[t_0 - \tau_1, b] \cap C^1[t_0, b],\ x_2 \in C[t_0 - \tau_2, b] \cap C^1[t_0, b]$.

From the above considerations, we can formulate the following theorem

Theorem 3.3. *Consider the problem (3.3)–(1.2), in the conditions (H_1)-(H_4). Then the problem (3.3)–(1.2) has a unique solution $(x_1^*, x_2^*),\ x_1^* \in C[t_0 - \tau_1, b] \cap C^1[t_0, b],\ x_2^* \in C[t_0 - \tau_2, b] \cap C^1[t_0, b]$ and the solution is differentiable on τ_1 and τ_2.*

Proof. In what follows we consider the following integral equations:

$$x_1(t, \tau_1, \tau_2) = \\ = \begin{cases} \varphi(t),\ t \in [t_0 - \tau_1, t_0], \\ \varphi(t_0) + \int_{t_0}^t f_1(s, x_1(s, \tau_1, \tau_2), x_2(s, \tau_1, \tau_2), x_1(s - \tau_1, \tau_1, \tau_2), x_2(s - \tau_2, \tau_1, \tau_2))ds,\ t \in [t_0, b], \end{cases}$$

$$x_2(t, \tau_1, \tau_2) = \\ = \begin{cases} \psi(t),\ t \in [t_0 - \tau_2, t_0], \\ \psi(t_0) + \int_{t_0}^t f_2(s, x_1(s, \tau_1, \tau_2), x_2(s, \tau_1, \tau_2), x_1(s - \tau_1, \tau_1, \tau_2), x_2(s - \tau_2, \tau_1, \tau_2))ds,\ t \in [t_0, b]. \end{cases} \tag{3.4}$$

Now, let take the operator

$$A_f : C[t_0-\tau_1,b]\times C[t_0-\tau_2,b]\to C[t_0-\tau_1,b]\times C[t_0-\tau_2,b],$$

given by the relation

$$A_f(x_1,x_2)=(A_{f_1}(x_1,x_2),A_{f_2}(x_1,x_2))$$

where

$$A_{f_1}(x_1,x_2)(t,\tau_1,\tau_2)=\begin{cases}\varphi(t),\ t\in[t_0-\tau_1,t_0]\\ \varphi(t_0)+\int_{t_0}^t f_1(s,x_1(s,\tau_1,\tau_2),x_2(s,\tau_1,\tau_2),\\ x_1(s-\tau_1,\tau_1,\tau_2),x_2(s-\tau_2,\tau_1,\tau_2))ds,\ t\in[t_0,b],\end{cases}$$

$$A_{f_2}(x_1,x_2)(t,\tau_1,\tau_2)=\begin{cases}\psi(t),\ t\in[t_0-\tau_2,t_0],\\ \psi(t_0)+\int_{t_0}^t f_2(s,x_1(s,\tau_1,\tau_2),x_2(s,\tau_1,\tau_2),\\ x_1(s-\tau_1,\tau_1,\tau_2),x_2(s-\tau_2,\tau_1,\tau_2))ds,\ t\in[t_0,b].\end{cases}$$

Let $X:=C[t_0-\tau_1,b]\times C[t_0-\tau_2,b]$ and $\|\cdot\|_C$, the Chebyshev norm on X. It is clear, from the proof of the Theorem 1 ([4]), that in the conditions (H_1)–(H_4), the operator A_f is a Picard operator.

Let (x_1^*,x_2^*) the only fixed point of A_f.

We consider the subset $X_1\subset X$,

$$X_1=\{(x_1,x_2)\in X \mid \frac{\partial x_i}{\partial t}\in C[t_0-\tau,b],\ i=1,2\}.$$

We remark that $(x_1^*,x_2^*)\in X_1,\ A(X_1)\subset X_1,\ A:(X_1,\|\cdot\|_C)\to(X_1,\|\cdot\|_C)$ is PO.

We suppose that there exists $\frac{\partial x_i^*}{\partial \tau_1}, \frac{\partial x_i^*}{\partial \tau_2},\ i=1,2.$

Then, from (3.4) we have that:

$$\begin{aligned}&\frac{\partial x_i^*(t,\tau_1)}{\partial \tau_1}=\\ &=\int_{t_0}^t \frac{\partial f_i(s,x_1^*(s,\tau_1),x_2^*(s,\tau_1),x_1^*(s-\tau_1,\tau_1),x_2^*(s-\tau_2,\tau_1))}{\partial u_1}\cdot\frac{\partial x_1^*(s,\tau_1)}{\partial \tau_1}ds+\\ &+\int_{t_0}^t \frac{\partial f_i(s,x_1^*(s,\tau_1),x_2^*(s,\tau_1),x_1^*(s-\tau_1,\tau_1),x_2^*(s-\tau_2,\tau_1))}{\partial u_2}\cdot\frac{\partial x_2^*(s,\tau_1)}{\partial \tau_1}ds+\\ &+\int_{t_0}^t \frac{\partial f_i(s,x_1^*(s,\tau_1),x_2^*(s,\tau_1),x_1^*(s-\tau_1,\tau_1),x_2^*(s-\tau_2,\tau_1))}{\partial u_3}\cdot\\ &\cdot\left[\frac{\partial x_1^*(s-\tau_1,\tau_1)}{\partial t}(-1)+\frac{\partial x_1^*(s-\tau_1,\tau_1)}{\partial \tau_1}\right]ds+\\ &+\int_{t_0}^t \frac{\partial f_i(s,x_1^*(s,\tau_1),x_2^*(s,\tau_1),x_1^*(s-\tau_1,\tau_1),x_2^*(s-\tau_2,\tau_1))}{\partial u_4}\cdot\frac{\partial x_2^*(s-\tau_2,\tau_1)}{\partial \tau_1}ds,\end{aligned}$$

where $t\in[t_0,b],\ i=1,2.$

This relation suggests us to consider the following operator

$$C_f : X\times X\to X$$

where

$$\begin{aligned} C_f(x_1,x_2,u,v)(t,\tau_1) &= 0, \text{ for all } t\in[t_0-\tau_2,t_0] \\ C_f(x_1,x_2,u,v)(t,\tau_1) &= 0, \text{ for all } t\in[t_0-\tau_1,t_0] \end{aligned}$$

and

$$\begin{aligned} & C_f(x_1,x_2,u,v)(t,\tau_1) := \\ & = \int_{t_0}^{t} \frac{\partial f_i(s,x_1(s,\tau_1),x_2(s,\tau_1),x_1(s-\tau_1,\tau_1),x_2(s-\tau_2,\tau_1))}{\partial u_1} u(s,\tau_1)ds + \\ & + \int_{t_0}^{t} \frac{\partial f_i(s,x_1(s,\tau_1),x_2(s,\tau_1),x_1(s-\tau_1,\tau_1),x_2(s-\tau_2,\tau_1))}{\partial u_2} v(s,\tau_1)ds + \\ & + \int_{t_0}^{t} \frac{\partial f_i(s,x_1(s,\tau_1),x_2(s,\tau_1),x_1(s-\tau_1,\tau_1),x_2(s-\tau_2,\tau_1))}{\partial u_3} \cdot \\ & \cdot \left[\overline{u}(s-\tau_1,\tau_1)\cdot(-1) - u(s-\tau_1,\tau_1)\right] ds + \\ & + \int_{t_0}^{t} \frac{\partial f_i(s,x_1(s,\tau_1),x_2(s,\tau_1),x_1(s-\tau_1,\tau_1),x_2(s-\tau_2,\tau_1))}{\partial u_4} v(s-\tau_2,\tau_1)ds, \end{aligned}$$

for all $t\in[t_0,b]$.

We denoted here

$$\begin{aligned} u(t) &= \frac{\partial x_1(t)}{\partial \tau_1},\ v(t)=\frac{\partial x_2(t)}{\partial \tau_1},\ \overline{u}(t-\tau_1)=\frac{\partial x_1(t-\tau_1)}{\partial t}, \\ u(t-\tau_1) &= \frac{\partial x_1(t-\tau_1)}{\partial \tau_1},\ v(t-\tau_2)=\frac{\partial x_2(t-\tau_2)}{\partial \tau_1}. \end{aligned}$$

In this way we have the triangular operator

$$D : X\times X \to X\times X$$

$$(x_1,x_2,u,v)\to (A_f(x_1,x_2), C_f(x_1,x_2,u,v))$$

where A_f is a Picard operator and $C_f(x_1,x_2,\cdot,\cdot): X\to X$ is an L-contraction, with $L=\dfrac{4L_f}{\rho}$, where ρ is the Bielecki constant we use in [4].

From the fibre contraction theorem we have that the operator D is Picard operator and $F_D=(x_1^*,x_2^*,u^*,v^*)$.

Let (x_1^*,x_2^*,u^*,v^*) the only fixed point of the operator D. Then the sequences

$$(x_{1,n+1},x_{2,n+1}) := A(x_{1,n},x_{2,n}),\ n\in\mathbb{N},$$
$$(u_{n+1},v_{n+1}) := C(x_{1,n},x_{2,n},u_n,v_n),\ n\in\mathbb{N},$$

converge uniformly (with respect to $t\in X$) to $(x_1^*,x_2^*,u^*,v^*)\in F_D$, for all $x_{1,0},x_{2,0},u_0,v_0\in X$.

If we take

$$\begin{aligned} x_{1,0} &= 0,\ x_{2,0}=0, \\ u_0 &= \frac{\partial x_{1,0}}{\partial \tau_1}=0,\ v_0=\frac{\partial x_{2,0}}{\partial \tau_1}=0, \end{aligned}$$

then

$$\begin{aligned} u_1 &= \frac{\partial x_{1,1}}{\partial \tau_1}, \\ v_1 &= \frac{\partial x_{2,1}}{\partial \tau_1}. \end{aligned}$$

By induction, we obtain that

$$\begin{aligned} u_n &= \frac{\partial x_{1,n}}{\partial \tau_1},\ \forall n \in \mathbb{N}, \\ v_n &= \frac{\partial x_{2,n}}{\partial \tau_1},\ \forall n \in \mathbb{N}. \end{aligned}$$

So

$$x_{1,n} \overset{unif}{\to} x_1^* \text{ as } n \to \infty,$$

$$x_{2,n} \overset{unif}{\to} x_2^* \text{ as } n \to \infty,$$

$$\frac{\partial x_{1,n}}{\partial \tau_1} \overset{unif}{\to} u^* \text{ as } n \to \infty,$$

$$\frac{\partial x_{2,n}}{\partial \tau_1} \overset{unif}{\to} v^* \text{ as } n \to \infty.$$

From the above consideration we have that there exist $\frac{\partial x_i^*}{\partial \tau_1},\ i = 1, 2$ and

$$\frac{\partial x_1^*}{\partial \tau_1} = u^*, \qquad \frac{\partial x_2^*}{\partial \tau_1} = v^*.$$

Analogously we can prove the differentiability with respect to τ_2. □

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